Math 100 Lecture: Antidifferentiation

Lecturer: Jose Maria L. Escaner IV, Ph.D.
Lecture 24

Antidifferentiation is the reverse process of differentiation. That is, an antiderivative of a function $f(x)$ with respect to x is a function $F(x)$ for which $F'(x) = f(x)$. Note that the antiderivative of a function is not entirely unique. For example, an antiderivative of $f(x) = 3x^2$ is $F(x) = x^3$, since $F'(x) = f(x)$. However, another antiderivative of $f(x)$ is $\tilde{F}(x) = 3x^2 + 3$ as its derivative will also give f. As matter of fact, any antiderivative of f takes the form $F(x) = x^3 + C$, where C is any arbitrary constant.

Theorem 1 *Antiderivatives of a function differ only by constant.*

Indeed, suppose F and G are antiderivatives of a function f. Then $F'(x) = f(x)$ and $G'(x) = f(x)$. Thus $D[F(x) - G(x)] = F'(x) - G'(x) = 0$. This would only mean that the rate of change of $F(x) - g(x)$ does not change and hence the difference is some constant C. That is, $F(x) - G(x) = C$.

Theorem 2 *If F is a particular antiderivative of f on an interval I, then every antiderivative of f on I is given by $F(x) + C$, where C is an arbitarary constant, and all antiderivatives of f on I can be obtained by assigning particular values to C.*

Notations: Since antidifferentiation is the inverse process of differentiation, one may write $D^{-1}[3x^2] = x^3 + C$. Leibniz introduced another notation to find the general antiderivative of a function. We write

$$\int f(x)dx = F(x) + C \qquad \text{where } F'(x) = f(x) \quad \text{and } d(F(x)) = f(x)dx.$$

In the expression, the symbol $\int$ denotes antidifferentiation, and the differential dx indicates the end of the expression to be antidifferentiated. dx also indicates that the expression must be antidifferentiated with respect to x. This is important especially when we have to deal with functions of more than one variable in the future.

Exercises: Intuitively, find the general antiderivative of the following functions:

1. $f(x) = 6x^2$
2. $f(x) = 4x^5$

We now present different theorems that will help us in our antidifferentiation.

Theorem 3 $\int dx = x + C$.

Theorem 4 *Antidifferentiation is a linear process. That is, if f and g are functions with antiderivatives $F(x)$ and $G(x)$ respectively, and k is any constant, then*

1. *$D^{-1}[f(x) \pm g(x)] = F(x) \pm G(x)$. Furthermore, this can be extended to a finite number of summands of functions.*
2. *$D^{-1}[cf(x)] = kF(x)$.*

Theorem 5 *If n is any rational number, then* $\int x^n dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C & n \neq 1 \\ \ln|x| + C & n = 1 \end{cases}$.

Theorem 6 *For any differentiable variable u, and $a > 0$,* $\int a^u \; du = \dfrac{a^u}{\ln a} + C$. *In particular,* $\int e^u \; du = e^u + C$.

Exercises: Perform the following antidifferentiation.

1. $\int (3x^2 + 2x + 1) \; dx$
2. $\int (3t^4 + 5t - 6) \; dt$
3. $\int (x^{5/2} - 2x^{3/2} + x^{1/2}) \; dx$
4. $\int \left(\dfrac{3}{x^4} + 2x^{2/3} - 2 \right) \; dx$
5. $\int \left(\sqrt[3]{x^2} + \dfrac{4}{\sqrt[4]{x^4}} \right) \; dx$
6. $\int 5\cos x - 3\sin x \; dx$
7. $\int \dfrac{\sin x}{\cos^2 x} \; dx$
8. $\int \dfrac{3\tan\theta - 4\cos^2\theta}{\cos\theta} \; d\theta$
9. $\int (e^{2x} - 3e^x + 2) \; dx$
10. $\int \dfrac{x^2 - 7x - 5}{x} \; dx$

Exercises: The following are exercises that deal with finding a particular solution of a differential equaiton. That is, finding a specific $y = f(x)$ from the general antiderivative given an initial value. Solve the following.

1. If $\dfrac{dy}{dx} = 2x + 1$ and $y(0) = 3$, then what is y?
2. If $\dfrac{dy}{dx} = x^4 - 3x + \dfrac{3}{x^{-3}}$ and $y(1) = -1$, then what is y?
3. The point (3,2) is on a curve, and at any point (x, y) on the curve the tangent line has a slope equal to $2x - 3$. Find the equation of the curve.
4. Vertical motions of objects near the earth's surface are affected mainly by the gravitational acceleration $g = -32 \; ft/s^2$ and possibly an initial velocity. If you throw a ball straight upward from the ground with an initial velocity of 96 ft/s, how high does the ball rise, and how long does it remain aloft?