Math 100 Lecture: Indeterminate Forms

Lecturer: Jose Maria L. Escaner IV, Ph.D.
Lecture 23

Definition 1 *The function f/g is said to be indeterminate (of the form $\frac{0}{0}$) at the number a if f and g are two functions with $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. The function f/g is said to be indeterminate (of the form $\pm\frac{\infty}{\infty}$) at the number a if f and g are two functions with $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.*

Some problems dealing dealing with indeterminate forms of the form $\frac{0}{0}$ have been introduced before, and ways to solve these problems have also been discussed. For example, the problem $\lim_{x\to 0} \dfrac{\sin x}{x}$ has been found to be equal to 1 using the squeeze theorem, despite the fact that the expression itself is indeterminate at 0. Also, the problem $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$ has been found to be equal to $\frac{1}{2}$ by rationalizing the numerator. Now, another method is presented to deal with indeterminate forms of the form $\frac{0}{0}$.

Theorem 1 (L'Hopital's Rule) *Suppose f/g is indeterminate of the form $\frac{0}{0}$ (or $\pm\frac{\infty}{\infty}$ at the number a. If $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = L$.*

The L'Hopital's Rule only work for these two indeterminate forms and may be used more than once for as long as it is indeterminate. However, the rule is not foolproof. That is, there is a possibility that one would not be able to get an answer however we use the rule.

Exercises: Evaluate the following limits.

1. $\lim_{x\to\infty} \dfrac{x^2-1}{2x^2+1}$
2. $\lim_{x\to\infty} \dfrac{\ln x}{x-1}$
3. $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$
4. $\lim_{x\to 0} \dfrac{x+\tan x}{\sin x}$
5. $\lim_{x\to\infty} \dfrac{e^x}{x^3}$
6. $\lim_{x\to 0} \dfrac{xe^x - x}{\sin^2 x}$
7. $\lim_{x\to 0} \dfrac{\tanh^{-1} 2x}{\tan^{-1} 2x}$

<u>Other indeterminate forms</u>

Other indeterminate forms are as follows: $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, and ∞^0. To solve these kinds of problems, one has to transform the expressions in the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that the L'Hopital's Rule may be used.

1. $\lim_{x\to 0^+} x\ln x$
2. $\lim_{x\to -\infty} x^2 e^x$
3. $\lim_{x\to\infty} e^{-x}\ln x$
4. $\lim_{x\to 0} \left(\dfrac{1}{x} - \csc x\right)$
5. $\lim_{x\to 0} (\csc x - \cot x)$
6. $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$
7. $\lim_{x\to 0^+} x^{\sin x}$
8. $\lim_{x\to 0^+} (\sin x)^{\tan x}$
9. $\lim_{x\to 0} (1-2x)^{1/x}$