Math 100 Lecture: Concavity, Points of Inflection and the Second Derivative Test

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Lecture 20

Definition 1 *The graph of a function f is said to be concave upward (downward)at the point $(c, f(c))$ if $f'(c)$ exists and if there is an open interval I containing c such that for all values of $x \neq c$ in I the point $(x, f(x))$ on the graph is above (below) the tangent line to the graph at $(c, f(c))$.*

If a function is either increasing on an interval or decreasing on an interval, then it is *monotonic* on the interval.

Theorem 1 *Let the function f be differentiable on some open interval I containing c.*

- *if $f''(c) > 0$ then the graph of f is concave upward at $(c, f(c))$.*
- *if $f''(c) < 0$ then the graph of f is concave downward at $(c, f(c))$.*

The proof of this theorem entails some analytic geometry and will be left as an exercise to the student. In the theorem, the only cases considered are $f''(c) > 0$ or $f''(c) < 0$. So what happens when $f''(c) = 0$? If there is a point on the graph of the function at which the concavity changes and the graph has a tangent line there, then the graph crosses its tangent line at this point.

Definition 2 *The point $(c, f(c))$ is a point of inflection of the graph of the function f if the graph has a tangent line there, and if there exists an open interval I containing c such that if x is in I, then either*

- *$f''(x) < 0$ if $x < c$, and $f''(x) > 0$ if $x > c$; or*
- *$f''(x) > 0$ if $x < c$, and $f''(x) < 0$ if $x > c$.*

Theorem 2 *Let f be a differentiable function on some open interval containing c, and $(c, f(c))$ be a point of inflection of the graph of f. Then if $f''(c)$ exists, then $f''(c) = 0$.*

The converse of the theorem is not true. For example, if $f(x) = x^4$, then $f''(0) = 0$. But the origin is not a point of inflection. As a matter of fact the graph of f is concave upward everywhere.

Theorem 3 (Second Derivative Test for Relative Extrema) *Let c be a critical number of a function f at which $f'(c) = 0$, and let f'' exist for all values of x in some open interval containing c,*

- *if $f''(c) > 0$, then f has a relative maximum value at c; and*
- *if $f''(c) < 0$, then f has a relative maximum value at c.*

To determine analytically the relative extrema and points of inflection, if any, of f:

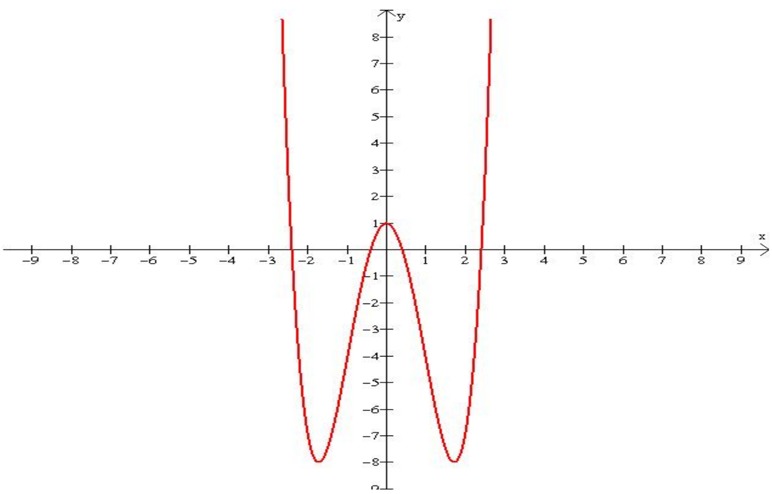
1. Get $f'(x)$ and $f''(x)$.
2. Determine all possible critical numbers of f.

3. Equate $f''(x)$ to zero and solve all possible points of inflection.

4. Apply the First and Second Derivative Tests.

Example: Find all relative extrema and points of inflection, if any, of the function $f(x) = x^4 - 6x^2 + 1$. Determine the intervals for which the graph is increasing/decreasing and concave upward/downward. Sketch the graph.
First, $f'(x) = 4x^3 - 12x$. Equating the derivative to zero, we get critical numbers $\pm\sqrt{3}$ and 0. Now $f''(x) = 12x^2 - 12$. Equating this to zero, the possible point of inflection may occur when $x = \pm 1$. We now construct the following table: The sketch of the graph is as shown below:

	$f(x)$	$f'(x)$	$f''(x)$	Conclusions
$x < -\sqrt{3}$		$-$	$+$	f is decreasing; f is concave upward
$x = -\sqrt{3}$	-8	0	$+$	f has a relative minimum value f is concave upward
$-\sqrt{3} < x < -1$		$+$	$+$	f is increasing f is concave upward
$x = -1$	-4	$+$	0	f is increasing f has a point of inflection
$-1 < x < 0$		$+$	$-$	f is increasing f is concave downward
$x = 0$	0	0	$-$	f has a relative maximum value f is concave downward
$0 < x < 1$		$-$	$-$	f is decreasing f is concave downward
$x = 1$	-4	$-$	0	f has a point of inflection f is concave downward
$1 < x < \sqrt{3}$		$-$	$+$	f is decreasing f is concave upward
$x = \sqrt{3}$	-8	0	$+$	f has a relative minimum value f is concave upward
$x > \sqrt{3}$		$+$	$+$	f is inreasing f is concave upward



Exercises: Find all relative extrema and point(s) of inflection of the function and sketch the graph. Make a table indicating where the function has a relative extrema, where it has an inflection, where it is decreasing/increasing, and where it is concave upward/downward.

1. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$
2. $f(x) = 4x^2 + 8x + 13$
3. $f(x) = x^3 - 12x + 17$
4. $f(x) = 3x^4 + 4x^3 - 12x^2$
5. $f(x) = x^3 - 6x^2 + 9x + 1$
6. $f(x) = x^3 - 3x^2 + 7x - 3$
7. $f(x) = (x + 2)(x - 2)^3$