Math 100 Lecture: Increasing and Decreasing Functions and the First Derivative Test

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Lecture 19

Definition 1 *A function f defined on an interval is increasing (decreasing) on the interval if and only if*

$$f(x_1) < f(x_2) \quad (f(x_1) < f(x_2)) \qquad \textit{whenever } x_1 < x_2$$

where x_1 and x_2 are any numbers in the interval.

If a function is either increasing on an interval or decreasing on an interval, then it is *monotonic* on the interval.

Theorem 1 *Let the function f be continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b).*

- *if $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$; else*
- *if $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$*

Proof: Let x_1 and x_2 be any numbers in $[a, b]$ such that $x_1 < x_2$.

Case 1: Suppose $f'(x) > 0$ for all $x \in (a, b)$. From the hypotheses given, there exists a number $c \in (a, b)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $x_1 < x_2$, then this means $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$.

Case 2: Suppose $f'(x) < 0$ for all $x \in (a, b)$. From the hypotheses given, there exists a number $c \in (a, b)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $x_1 < x_2$, then this means $f(x_2) - f(x_1) < 0$ or $f(x_1) > f(x_2)$.

End of Proof.

Theorem 2 (First Derivative Test for Relative Extrema) *Let the function f be continuous at all points of the open interval (a, b) containing the number c, and suppose f' exists at all points in (a, b) except possible at c.*

- *if $f'(x) > 0$ for all values of x in some open interval having c as its right endpoint, and $f'(x) < 0$ for all values of x in some open interval having c as its left endpoint, then f has a relative maximum value at c; and*
- *if $f'(x) < 0$ for all values of x in some open interval having c as its right endpoint, and $f'(x) > 0$ for all values of x in some open interval having c as its left endpoint, then f has a relative maximum value at c.*

To determine analytically the relative extrema of f:

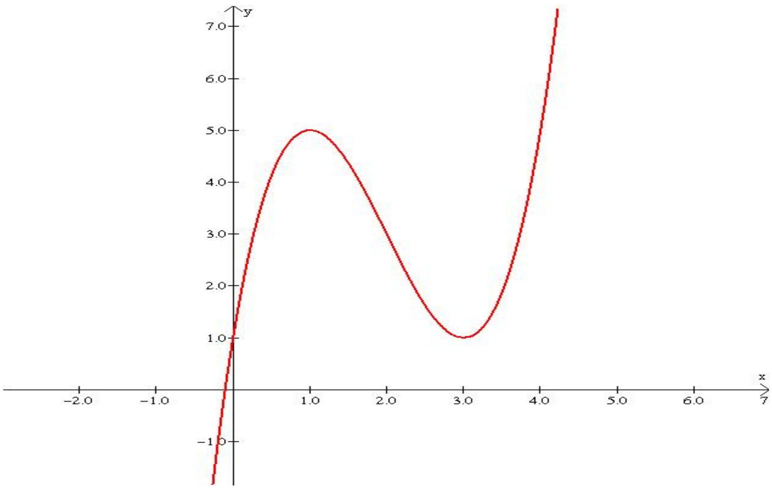
1. Get $f'(x)$.
2. Determine all possible critical numbers of f.

3. Apply the First Derivative Test.

In sketching the graphs, it is also good to know the roots of the function, if there are any. We now present an example.

Example: Find all relative extrema of the function $f(x) = x^3 - 6x^2 + 9x + 1$ and sketch the graph. First, $f'(x) = 3x^2 - 12x + 9$. Equating the derivative to zero, we get critical numbers 3 and 1. We now construct the following table: The sketch of the graph is as shown below:

	$f(x)$	$f'(x)$	Conclusions
$x < 1$		$+$	f is increasing
$x = 1$	5	0	f has a relative maximum value
$1 < x < 3$		$-$	f is decreasing
$x = 3$	1	0	f has a relative minimum value
$x > 3$		$+$	f is increasing



Exercises: Find all relative extrema of the function and sketch the graph.

1. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$
2. $f(x) = 4x^2 + 8x + 13$
3. $f(x) = x^3 - 12x + 17$
4. $f(x) = 3x^4 + 4x^3 - 12x^2$