Math 100 Lecture: Rectilinear Motion and Application of Derivatives in Econonomics

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Lecture 16

Recall that the definition of the derivative of a function at a point evolved from the slope a tangent line on a curve at a point, which is the limiting case of slope of a secant line on a curve. In a sense, what has become a usual rate of change of functional values became the *instantaneous* rate of change of the function at a number. We shall adapt this technique (in a different jargon) to give us a more practical application of the concept of derivatives.

1 Rectilinear Motion of a Particle

We consider now a simple application in physics: rectilinear motion. That is, the motion of a particle on a (horizontal) line. Let us assume that the right direction is considered as the postive direction while the other direction gives a negative direction. Thus if a particle from an arbitrary point of origin moves to the right direction, the particle traverses a positive distance s. If t is the time (in seconds) then the **displacement** function from the origin at a particular instant t is given by $s = s(t)$. That is, the directed distance depends on the time the particle travels from the origin to a point on the line.

Now, suppose a particle from the origin moves at a distance to the right. Then after t_0 seconds, it has traversed $s(t_0)$ units of distance. From this point, the particle moves on either direction for h more seconds. Then *averate rate of change* in the distance, or better known as the average velocity, between time $[t_0, t_0 + h]$ is given by

$$\text{velocity} = \frac{s(t_0 + h) - s(t_0)}{(t_0 + h) - t_0} = \frac{s(t_0 + h) - s(t_0)}{h}.$$

If we now let h approach zero, then we get the **instantaneous** velocity of the particle, v, at time t_0 by

$$v(t_0) = s'(t_0) = \frac{d}{dt}s(t_0) = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}.$$

In general, the instantaneous velocity of the displacement function at any time t is the first derivative with respect to t of the displacement function, i.e., $v(t) = s'(t)$.

The instantaneous velocity may either be positive or negative depending on how the particle moved on the horizonal line. Note that when the instantaneous velocity is zero, the particle is at rest. One important term also included in rectilinear motion is speed. **Speed** is defined as the absolute value of the instantaneous velocity. Thus the instantaneous velocity is a directional (vector) parameter while the speed is a scalar parameter.

The second derivative of the displacement function, $a(t) = s''(t)$ is called the instantaneous **acceleration** of the particle. It is so termed since it is the instantaneous rate of change in the velocity of the particle.

Exercises: A particle is moving along a horizontal line according to the given equation $s(t)$ where s is measured in meters and t is measured in seconds. Find, if they exist, the displacement $s(t)$, the intantaneous velocity $v(t)$ and the instantaneous acceleration $a(t)$ at the given particular t_1.

1. $s(t) = 4t^3 - 3t^2 + 5t - 1$, $t_1 = 0$
2. $s(t) = \dfrac{2}{t-1}$, $t_1 = 2$
3. $s(t) = -16t^2 + 32t + 12$, $t_1 = 4$
4. $s(t) = cos3t - sin2t$, $t_1 = 2$

Exercises: Solve the following problems.

1. Free-falling bodies: A ball is dropped from a height of 30 feet from the horizontal ground. The height h in feet at any time t seconds is given by $h(t) = -16t^2 + 30$. Find
   (a) the average velocity from $t = 1$ to $t = 2$;
   (b) the instantaneous velocity at $t = 1$ and at $t = 2$;
   (c) the time when ball reaches the ground; and
   (d) the the velocity of the ball as it reaches the ground.
2. Curvilinear motion: A ball is thrown vertically upward from the ground with an initial velocity of 20 feet per second. If the positive direction of the distance from the starting point is up, the equation of the motion is given by $s(t) = -16t^2 + 20t$.
   (a) the instantaneous velocity at $t = 3$;
   (b) the time for the ball to reach the highest point;
   (c) the total time for the ball to reach the ground; and
   (d) the instantaneous velocity as the ball reaches the ground.
3. Curvilinear motion: Carlo throws a ball upward from a window in a building 20 feet above the ground. If the initial velocity is 13 feet per second, the equation of the motion is given by $s(t) = -16t^2 + 13t + 20$. Find
   (a) the instantaneous velocity at $t = 1$;
   (b) the time for the ball to reach the highest point;
   (c) the total time for the ball to reach the ground; and
   (d) the instantaneous velocity as the ball reaches the ground.

2 Marginal Revenue, Cost and Profit Functions

An application of the derivatives in business and economics is the marginal cost/revenue of a cost/revenue function. Simply put, a cost/revenue function is a function that determines the cost/revenue of a product/service based on some independent factor such as the number of units produced, number of hours served, and the like. Thus the cost/revenue varies with the factor involved. The cost function is usually denoted by $C(x)$ or $C(t)$, while the revenue function is denoted by $R(x)$ or $R(t)$, depending on the description of the independent factor.

The average cost/revenue of a product/service with respect to some interval is computed as the ratio between the difference of the costs/revenues in-between intervals and the difference of the intervals. This is the same as getting the slope of a (secant) line or getting the average velocity of a moving particle. When the difference is very small such that it approaches to zero, we get what is called the **marginal cost/revenue**. The marginal cost $C'(x)$ at $x = x_0$ gives the average cost per unit when x_0 units are made. Similarly, the marginal revenue $R'(x)$ at $x = x_0$ gives the average price price per unit when x_0 units are sold.

The net profit function $P(x)$ is the difference between the revenue function and the cost function. That is, $P(x) = R(x) - C(x)$. Taking the derivative of the equation gives the marginal net profit function $P'(x)$.

Exercises:

1. The total revenue received from sale of x desks is $R(x)$ dollars, and $R(x) = 200x - \frac{1}{3}x^2$. Find (a) the marginal revenue function; (b) the marginal revenue when $x = 30$; (c) the actual revenue from the sale of the 31st desk.
2. The total cost of manufacturing x watches in a certain plant is given by $C(x) = 1500 + 3x^2$. Find (a) the marginal cost function; (b) the marginal cost when $x = 40$; and (c) the actual cost of manufacturing the 41st watch.

As an ending note, other problems include population logistics, volumes, areas, for as long as it involves a rate of change in the unknown quantity.