THE BASIC MATH OF DEFAULT-MODE CREDIT LOSS DISTRIBUTIONS

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The purpose of this note is to summarize some of the math used by the credit rating agencies to calculate loss distributions for structured finance transaction collateral pools. It is meant only to provide a very high-level view, and references are provided for those wishing to delve deeper into the various methodologies.

The Binomial Distribution and Uncorrelated Defaults

If the portfolio consists of equally-sized positions, all with the same probability of default (*PD*) and loss given default (*LGD*), a straight binomial distribution can be used to generate the probability distribution of losses. In this regard, the probability associated with the *K* losses in an *N*-obligor portfolio:

$$P_{N,K} = \frac{N!}{K!(N-K)!} PD^K (1-PD)^{N-K}$$

For each of the $N+1$ default scenarios the loss amount is: $L_J = \overline{F} \times LGD \times K$ where $\overline{F}$ is the size of the individual position. Some of the key descriptive statistics include:

- Expected loss = $EL = PD \times LGD$
- Probability of loss = $PL = 1-(1-PD)^N$
- Average loss severity = $ALS = EL/PL$
- Unexpected loss = $UL = \sqrt{\sum_{K=0}^{N} P_{N,K} PD^K (1-PD)^{N-K} LGD^2 (K/N-PD)^2}$
- Value at Risk (VaR) at $1\text{-}\alpha$ confidence level = $VaR_\alpha = L_Z$ where Z is set at point where $\sum_{K=0}^{Z} P_K \geq 1-\alpha$

Table 1 shows some risk measures for a portfolio of uncorrelated AAA exposures of increasing diversification (from one to sixty obligors[1]). It is noteworthy that, while *EL* is fixed at 0.000055%, *PD* rises with the number of obligors (N). However, the increasing *PD*s are offset by the decreasing *ALS* (last column) as the portfolio becomes more diverse, which is why the *UL* decreases. Although not tabulated, the *VaR* is zero for any reasonable confidence interval, due to the extremely high quality of the underlying credits. For example, when there are sixty obligors, the likelihood of not losing anything at all is 99.994%.

Anyways, the point of Table 1 is that, although it is obvious that risk decreases in diversity, the *EL* statistic does not pick this up, and the *PD* perversely increases. This perversion may be related to the well-known problem of VaR's non-additivity.[2] It would appear that *UL*s and severities are more consistent. Hence, *UL*, or a combination of *EL* and *UL* are probably superior metrics.

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1 The individual default probabilities in the AAA portfolio are all (P=) 0.0001% and the assumed loss given default (LGD) is 55%. See Appendix A for a Moody's tabulation of default probabilities.

2 Artzner *et al* (1999) show that VaR is not "sub-additive" and conclude that "the use of value at risk does not encourage and, indeed, sometimes prohibits diversification…". See also Frey & McNeil (2002) and O'Kane & Schloegl (2002).

Table 1: Risk Measures for Portfolio of Uncorrelated AAA Exposures (One-year horizon, default mode, and all holdings have equal sizes and PDs) (All measures are expressed as a percent of total par value)				
Number of Obligors	**Expected Loss**	**Default Probability**	**Unexpected Loss**	**Average loss severity**
1	0.000055%	0.0001%	0.055000%	55.00%
2		0.0002%	0.038891%	27.50%
3		0.0003%	0.031754%	18.33%
4		0.0004%	0.027500%	13.75%
5		0.0005%	0.024597%	11.00%
10		0.0010%	0.017393%	5.50%
30		0.0030%	0.010042%	1.83%
60		0.0060%	0.007100%	0.92%

Correlated Defaults and Moody's "Binomial Expansion Technique (BET) Approach

The BET method relies on the use of a simple diversification measure, the "diversity score" that is used to map the actual portfolio into a hypothetical portfolio of homogeneous assets. Hence, for the purpose of analysis, the actual portfolio is replaced with a much simpler hypothetical portfolio of homogeneous, uncorrelated assets. The number of assets in the hypothetical pool is assumed to equal the diversity score (D). Appendix B provides more detail on the mathematical foundations of the diversity score calculation. Given the assumption of uncorrelated defaults, the behaviour of this homogenous pool of assets can then be described by $D+1$ default scenarios, the probability of which can be calculated using the binomial formula. The expected losses for the mimicking portfolio are then determined based on these BET probabilities.

For rating multi-sector structured finance products, Moody's calculates the diversity score as follows for a pool of n assets:[3]

$$D = \frac{\left(\sum_{i=1}^{n} PD_i F_i\right)\left(\sum_{i=1}^{n}(1-PD_i)F_i\right)}{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\left(PD_i(1-PD_i)PD_j(1-PD_j)\right)^{1/2}F_iF_j}$$

where F_i is the size of the ith holding, PD_i is the probability of default associated with the ith asset, and ρ_{ij} is the <u>default</u> correlation[4] between assets i and j. This can be simplified by making various assumptions about the uniformity of holding sizes (F_i constant for all i), and default probabilities (PD_i the same for all i). Further streamlining can be achieved by assuming that all intra-sector pairwise default correlations are equal to ρ_{int} and all inter-sector pairwise default correlations are ρ_{ext}. If there are m sectors (Moody's distinguishes 33 different industry categories) and there are n_k in the kth sector:

$$D = \frac{n^2}{n + \rho_{ext} n(n-1) + (\rho_{\text{int}} - \rho_{ext})\sum_{k=1}^{m} n_k(n_k - 1)}$$

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3 See Appendix II of Gluck and Remeza (2000).

4 Appendix C discusses some of the nuances of asset versus default correlations.

Table 2 shows how dramatically diversity scores can decrease as correlations increase. The results in the table focus on an "ultra-homogeneous" portfolio in which all the default probabilities are equal, there are 10 sectors, each containing 6 equally-sized ($10mm) asset holdings. The table shows, for example, that if both the inter- and intra-sector pairwise correlations are both 5%, the diversity score is 15. However, when both correlations are equal to 20%, the score is 5.

<table>
<tr><th colspan="9">Table 2: Impact of alternative default correlations on diversity scores</th></tr>
<tr><th rowspan="2"></th><th rowspan="2">Inter-sector
Default correlation</th><th colspan="7">Intra-sector default correlation</th></tr>
<tr><th>0%</th><th>5%</th><th>10%</th><th>15%</th><th>20%</th><th>25%</th><th>30%</th></tr>
<tr><td rowspan="7">Sectors (m) = 10
Holding (Favg) = 10
Holdings per sector = 6</td><td>0%</td><td>60</td><td>48</td><td>40</td><td>34</td><td>30</td><td>27</td><td>24</td></tr>
<tr><td>5%</td><td>16</td><td>15</td><td>14</td><td colspan="2">13</td><td colspan="2">12</td></tr>
<tr><td>10%</td><td colspan="3">9</td><td colspan="4">8</td></tr>
<tr><td>15%</td><td>7</td><td colspan="6">6</td></tr>
<tr><td>20%</td><td colspan="7">5</td></tr>
<tr><td>25%</td><td colspan="7">4</td></tr>
<tr><td>30%</td><td colspan="7">3</td></tr>
</table>

For each of the $D+1$ scenarios a loss amount (L_K) is calculated as it was before, and the probability associated with the Kth scenario is $P_{D,K}$ where PD is equal to the average of the obligor-specific default probabilities in the portfolio.

Moody's uses this simple model when the portfolio is fairly homogeneous in terms of size of holdings and default probability distributions. When the pool is made up of two or more uncorrelated groups of assets having "markedly different average properties" Moody's may use the "multiple BET" (MBET).[5]

Correlated Defaults and Moody's "Correlated Binomial" (CB) Approach

The CB approach, introduced in Witt (2004), incorporates default correlations in an even more explicit and accurate way than does the BET. Moody's uses the CB approach for cash flow CDOs backed by pools of highly correlated assets that have low traditional diversity scores (e.g., below 10). It starts with the calculation of a "correlated diversity score" (D_C) which is a function of the D:

$$D_C = \frac{D(1-\rho)}{1-\rho D} \text{ and if all } n \text{ assets are identical } D_C = n$$

The Witt (2004) paper then goes on to show that the probability of K losses in a portfolio consisting of $N=D_C$ credits is:

$$P_{N,K} = \frac{N!}{K!(N-K)!}\sum_{J=0}^{N-K}(-1)^J \frac{(N-K)!}{J!(N-K-J)!}\prod_{i=1}^{J+K} PD_{i|i-1}$$

5 See Cifuentes and Wilcox (1998) For the extension from double to multiple BET, see Yoshizawa (2003). For a Moody's methodology "map" see Table 1 in Debuysscher and Szego (2003). Basically the BET and MBET assume zero inter-sector/pool correlations.

where $PD_{i|i-1}$ is the probability of i defaults conditional on $i-1$ defaults having occurred. The Witt (2004) paper provides the full VBA code to implement the methodology. A big chunk of the code is devoted to enabling the extremely high level of internal calculation precision required to solve some of the factorial expressions. The algorithm relies on the following key relationships and assumptions:

1. $PD_{J+1|J} = PD_{J|J-1} + (1 - PD_{J|J-1})\rho$ for all $j=1,2,\ldots,N-1$. This makes the conditional probability increase as other defaults occur, and is one of the factors that gives the CB the fatter tails that are associated with correlated variables. When this is solved sequentially, the expression simplifies down to $PD_{J|J-1} = 1-(1-PD)(1-\rho)^{J-1}$
2. $P_{K,K} = \prod_{J=1}^{K} PD_{J|J-1}$ (The probability of K defaults in a pool of K uncorrelated assets)
3. $P_{J-1,K} = P_{J-1,K-1} - P_{J,K}$ and $P_{1,1} = PD_{1|0} = PD$

Hence, the basic algorithm is as follows:

1. $P_{0,1} = 1 - P_{1,1}$
2. For K=2 to N:
   a. $PD_{K|K-1} = PD_{K-1|K-2}(1-\rho) + \rho$
   b. $P_{K,K} = PD_{K|K-1} P_{K-1,K-1}$
   c. For J=1 to 0 step -1: $P_{J,K} = P_{J,K-1} - P_{J+1,K}$
3. For J=0 to N: $P_{N,K} = \dfrac{N!}{K!(N-K)!} P_{K,N}$ (this last step is necessary because the K=2 to N loop calculates the probability of J defaults of N assets in any order…)

A Simplified "Structural" Credit Risk Model

The Monte Carlo (MC) approach to building and analyzing loss distributions is based on a highly simplified variation of a "structural" credit risk model. At the individual obligor level, it is assumed that default occurs when the value of the firm's assets (V_i) fall below some threshold level K_i, which in the standard literature is assumed to be the book value of the firm's debt. Hence, the loss dynamics can be described as follows:

$$L_i = \max\{K_i - V_i\} = if\{V_i < K_i; K_i - V_i; 0\}$$

To facilitate the implementation of the model, it is conventionally assumed that the log of the proportional change in the value of the underlying assets is normally distributed with $\mu_i \tau$ mean and $\sigma_i \sqrt{\tau}$ standard deviation, where τ is the time during which the change is evaluated (in years):

$$\ln(V_i + \Delta V_i) - \ln(V_i) \sim \Phi\{\mu_i \tau, \sigma_i \sqrt{\tau}\} \Leftrightarrow V_i + \Delta V_i = V_i \exp\{\mu_i \tau + \sigma_i \sqrt{\tau} \tilde{X}_i\}$$
$$\text{where } \tilde{X}_i \sim iid\ \Phi\{0,1\}$$

where $\Phi\{\mu, \sigma\}$ is the cumulative normal distribution. In this context, default occurs when $\ln(V_i + \Delta V_i) < \ln(K_i)$:

$$\tilde{X}_i < \frac{\ln(K_i / V_i) - \mu_i \tau}{\sigma_i \sqrt{\tau}} \quad \therefore \; PD_i = \Phi\{-DD_i\} \text{ where } DD_i = \frac{\ln(V_i / K_i) + \mu_i \tau}{\sigma_i \sqrt{\tau}}$$

PD_i is the probability of default, and DD_i is the "distance to default" measured in terms of standard deviations. The "rating based" methodologies invert the expression above to infer a DD_i from the default probability (PD_i) associated with the obligor's credit rating: $DD_i = \Phi^{-1}(PD_i)$ where $\Phi^{-1}(x)$ is the inverse of the standard normal distribution. Also, the expected loss inferred from this model is given below:

$$E\{\tilde{L}_i\} = PD_i E\{L\tilde{G}D_i\} \text{ where } E\{L\tilde{G}D_i\} = K_i - V_i \exp\{(\mu_i + \sigma_i^2 / 2)\tau\}\Phi\{-DD_i - \sigma_i\sqrt{\tau}\}/ PD_i$$

Of course, the μ and σ are rather critical parameters.[6] In the context of a risk neutral valuation framework, μ is a function of the so-called risk-free rate (r) and a risk aversion parameter (λ) with the σ parameter representing a "business risk" metric. (i.e., $\mu_i = r + \lambda\sigma$) In "generic" work, the relationship between K and V is often expressed as a leverage ratio (K/V) that represents a "financial risk" metric.

The Monte Carlo Approach[7]

The MC approach involves, for each simulation run, drawing a correlated standard normal random variable for each obligor in the portfolio, which is taken to represent the standardized change in the obligor's asset value over the appropriate horizon (ΔX_i). If $\Delta X_i < -DD_i$ a default is indicated and a loss or recovery is drawn from an appropriate distribution. For each simulation run, the default losses are accumulated for the n assets in the pool to arrive at a total loss (L_T):

$$L_T = \sum_{i=1}^{n} (\Delta X_i < -DD_i) PAR_i (1 - RR_i)$$

where PAR_i is the ith obligor par value and RR_i its recovery rate.

The correlated normal variables are generated from a Cholesky decomposition of the correlation matrix (see Appendix D), which is the same methodology, used by S&P and Fitch to rate collateralised debt obligations. For simulating the correlated standard normal variables, it is important that the correlation of the underlying assets be used, as opposed to equity or default correlations.

A step-by-step MC approach, with Excel/Poptools[8] functions given where appropriate, is presented below. For more detail and other options, like using a student-t distribution (instead of a normal distribution) see Glasserman (2004).

1. Determine the default probability (PD_i) for each of the N assets in the portfolio
2. Compute the default threshold for each of the N assets:
   $DD_i = \Phi^{-1}\{PD_i\} = NormSInv(PD_i)$

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6 The $\sigma^2/2$ appears in the LGD formula because while μ is the mean of the single-period proportional change, according to the properties of the lognormal distribution, the expected terminal proportional change is $\mu+\sigma^2/2$.

7 For a general description of an MC approach applied to the analysis of collateralized debt obligations, see Morokoff (2003).

8 The freeware Poptools Excel add-in package can be downloaded at http://www.cse.csiro.au/poptools. The Poptools functions used here are *dNormalDev(0,1)* which generates uncorrelated standard normal random numbers, and *Cholesky()* which performs a Cholesky decomposition.

3. Generate a vector of N uncorrelated standard-normally-distributed random variables: $U_i = NormSInv(Rand())$ or $U_i = dNormalDev(0,1)$ using the more accurate Poptools function.

4. Transform the vector of N uncorrelated random variables (**U**) into a vector of N correlated random variables (**X**) by multiplying them by the Cholesky decomposition of the NxN asset correlation matrix. If **C** is the NxN correlation matrix, the Cholesky decomposition is the NxN symmetric positive definite lower triangular matrix **A**, such that $\mathbf{C}=\mathbf{A}\mathbf{A}^T$. A lower triangular matrix has zeros on the upper right corners above the diagonal. (The superscript "T" denotes the "transpose" of the matrix. $\mathbf{A}^T$ is a matrix that has the first row of **A** as its first column, the second row of **A** as its second column… and the *n*th row of **A** as its *n*th column.) In Excel/Poptools terms:

$$(X_1, X_2 ... X_N) = MMULT(Cholesky(\Omega), (U_1, U_2 ... U_N))$$

   As for where the correlations are actually determined, there are several options. Appendix D discusses the Basel 2 approach, which prescribes PD-driven asset correlations that range from 12% to 24%. Alternatively, one could follow the leads of the Moody's and S&P who use fairly simplistic asset correlation assumptions. For example, S&P prescribes 30% intra-sector, and zero inter-sector correlations. Moody's will be assuming 15% intra-sector, and 3% inter-sector correlations. For references to more sophisticated approaches, including using equity return correlations, see de Servigny and Renault (2003).

5. Determine which assets default by comparing each element of the vector of correlated random numbers ($X_1, X_2 … X_N$) with the corresponding default threshold ($DD_1, DD_2 … DD_N$). When the random variable is less than the threshold (i.e., $X_i < DD_i$) a default loss is registered. The loss-given-default for each default event (LGD_i) can be determined in a number of ways:

   a. A constant proportion of the par value like, for example, the 45% prescribed by Basel 2 for senior unsecured claims and 75% for subordinated claims. One could also use LGDs based on rating agency historical data. For example, according to the most recent survey by Moody's (covering 1982 to 2003) the average LGD was 55.6% for senior unsecured bonds and 70.8% for senior subordinated bonds. The Moody's survey also breaks the data down according to instrument type (bonds versus loans) and industry. See Schuermann (2004) and the references therein, for a more extensive survey of empirical LGD data.

   b. Allow for uncertainty in LGDs, the impact of which can be quite significant for some structured finance transactions according to Goldbaum *et al* (2002). See Altman *et al* (2003b) for a review of the literature on stochastic LGDs, and Altman *et al* (2003a) where they show that using static LGDs underestimates credit risk. They also show, both theoretically and empirically, that default and LGDs are positively correlated. In models that do account for stochastic LGDs, beta distribution sampling is most common (see Ramaswamy (2004)) but this does not account for the correlated defaults and LGDs. Benzschawel (2003) develops a beta distribution sampling approach that does incorporate these correlations. The need to incorporate stochastic LGDs might depend on the quality of the starting portfolio. For a typical central bank portfolio of ultra-high-grade credits, using stochastic LGDs may not be worth the effort. For lower-grade portfolios, it probably is, since the default point would be likely to be hit more often.

6. Aggregate the individual losses and save the result (e.g., as $LOSS_j$ for $j=1$ to M, where M is the number of simulation runs).

7. Repeat steps 3-6 as many times as deemed appropriate (e.g., M times) and when finished sort the losses (i.e., $LOSS_1$, $LOSS_2 \ldots LOSS_M$) in ascending order. This sorting will become important for the value at risk (VaR) calculation.

8. Calculate the descriptive statistics:

   a. Expected loss = $EL = \sum_{j=1}^{M} LOSS_j / M$

   b. Unexpected loss = $UL = \sqrt{\sum_{j=1}^{M} LOSS_j^2 / M - EL^2}$

   c. Loss probability = $PL = \sum_{j=1}^{M} \left(LOSS_j > 0\right) / M$

   d. Average loss severity = $ALS = EL/PL$

   e. Value at risk at $1-\alpha$ confidence level = $VaR_\alpha = LOSS_Q$ where $Q = ROUND\left(M\left(1-\alpha\right),0\right)$

   f. Expected loss = $ES_\alpha = \dfrac{\sum_{j=1}^{M}\left(\left(LOSS_j > VaR_\alpha\right)\left(LOSS_l - VaR_\alpha\right)\right)}{\sum_{j=1}^{M}\left(LOSS_j > VaR_\alpha\right)}$

References

Altman, E., B. Brady, A. Resti and A. Sironi (2003a) *The link between default and recovery rates: theory, empirical evidence and implications*. NYU Stern School of Business working paper No. S-CDM-03-02, March.

Altman, E., A. Resti and A. Sironi (2003b) *Default recovery rates in credit risk modeling: a review of the literature and empirical evidence*. NYU Stern School of Business working paper No. S-CDM-03-11, December.

Artzner, P., Delbaen, F., Eber, J. and D. Heath (1999) "Coherent measures of risk." *Mathematical Finance* July 1999, 203-228.

Benzshawel, T. (2003) "A stochastic model for loss given default," Citigroup Global Markets *Quantitative Credit Analyst* October, 30-49.

Cifuentes, A and C Wilcox (1998): *The double binomial method and its application to a special case of CBO structures*. Moody's Investors Service Structured Finance Special Report, March.

Frey, R. and A.J. McNeil (2002) "VaR and expected shortfall in portfolios of dependent credit risks: conceptual and practical insights." *Journal of Banking & Finance* 26, 1317-1334.

Debuysscher, A and M Szego (2003: "*The Fourier transform method – overview*", Moody's Investors Service Structured Finance Special Report, January.

de Servigny, A. and O. Renault (2003) "Correlation evidence", *RISK* July, pp 90-94.

Fender, I. and J. Kiff (2004) *CDO rating methodology: some thoughts on accuracy, model risk and ratings shopping*. Unpublished draft, July.

Gersbach, H. and A. Lipponer (2003) "Firm defaults and the correlation effect." *European Financial Management*, vol. 9, 361-77.

Gibson, M.S. (2004) *Understanding the risks of synthetic CDOs.* US Federal Reserve Board FEDS paper No. 2004-36, July.

Glasserman, P. (2004) *Monte Carlo methods in financial engineering*. Springer-Varlag.

Gluck, J and H Remeza (2000): *Moody's approach to rating multi-sector CDOs*. Moody's Investors Service Structured Finance Special Report, September.

Goldbaum, L., G. Levington and P. Mazataud (2002) *Modelling recovery rates in European CDOs*. Moody's International Structured Finance Special Report, August 28.

Hull, J. and A. White (2004) *Valuation of a CDO and an nth to default CDS without Monte Carlo simulation*. Rotman School of Management, University of Toronto unpublished mimeo, February.

Jewell, J and M Livingston (1999): "A comparison of bond ratings from Moody's, S&P and Fitch IBCA", *Financial Markets, Institutions and Instruments*, vol 8(4), pp 1-45.

Morokoff, W.J. (2003) *Simulation methods for risk analysis of collateralized debt obligations*. Paper presented at the 2003 Winter Simulation Conference, December 7.

O'Kane, D. and L. Schloegl (2002) *Coherent risk measures applied to the default risk of credit portfolios and CDO tranches.* Lehman Brothers Fixed Income Quantitative Credit Research, January 31.

Pykhtin, M. and A. Dev (2003) "Course-grained CDOs." *RISK* January, pp 113-116.

Ramaswamy, S. (2004) *Managing credit risk in corporate bond portfolios*. John Wiley & Sons.

Schuermann, T. (2004) "What do we know about loss given default", in D. Shimko (ed.) (2004) *Credit Risk Models and Management*. 2nd edition, Risk Books, London.

Witt, G. (2004) *Moody's correlated binomial default distribution.* Moody's Investors Service Structured Finance Rating Methodology, August 10.

Yoshizawa, Y. (2003) *Moody's approach to rating synthetic CDOs*. Moody's Investors Service Structured Finance Special Report, July 29.

Zhou, C. (2001) "An analysis of default correlations and multiple defaults." *Review of Financial Studies*, vol. 14, No. 2, 555-76.

Appendix A: Basic Binomial Model Inputs

In order to calculate the expected probability of default (PD) the following steps are followed:

- Calculate the par value weighted average life (WAL) and rating (WAR) of the portfolio.
- Translate the WAR into an expected PD using the PDs in Table X. For example, if the WAL is one year and the WAR is "BB-" the PD is 2.81%. If the WAL is ten years the "BB-" PD is 17.66%.

Table X: Moody's Idealized PDs by WAR and WAL

Moody's Scale	**S&P/Fitch Scale**	**Weighted Average Life (WAL)**				
		1-year	**2-year**	**3-year**	**5-year**	**10-year**
Aaa	AAA	0.0001	0.0002	0.0007	0.0029	0.0100
Aa1	AA+	0.0006	0.0030	0.0100	0.0310	0.1000
Aa2	AA	0.0014	0.0080	0.0260	0.0680	0.2000
Aa3	AA-	0.0030	0.1900	0.0590	0.1420	0.4000
A1	A+	0.0058	0.0370	0.1170	0.2610	0.7000
A2	A	0.0109	0.0700	0.2220	0.4670	1.2000
A3	A-	0.0389	0.1500	0.3600	0.7300	1.8000
Baa1	BBB+	0.0900	0.2800	0.5600	1.1000	2.6000
Baa2	BBB	0.1700	0.4700	0.8300	1.5800	3.6000
Baa3	BBB-	0.4200	1.0500	1.7100	3.0500	6.1000
Ba1	BB+	0.8700	2.0200	3.1300	5.2800	9.4000
Ba2	BB	1.5600	3.4700	5.1800	8.4100	13.5000
Ba3	BB-	2.8100	5.5100	7.8700	11.8600	17.6600
B1	B+	4.6800	8.3800	11.5800	16.1200	22.2000
B2	B	7.1600	11.6700	15.5500	20.7100	27.2000
B3	B-	11.6200	16.6100	21.0300	27.0500	34.9000
Caa1	CCC+	17.381	22.2341	28.6386	36.3137	47.7000
Caa2	CCC	26.0000	32.5000	39.0000	48.7500	65.0000
Caa3	CCC-	50.9902	57.0088	62.4500	69.8212	80.7000
Ca	CC					
C	C					

Appendix B: Moody's Alternative Diversity Score Derivation[9]

The diversity score is intended to represent the number of independent, identical assets that poses the same loss distribution as the actual collateral portfolio. In a transaction with credit enhancement, such as that provided by the tranching of a CDO, we are primarily interested in the left tail of the return distribution (where losses to the rated tranches may occur). Thus, we want to closely approximate this tail.

One means of calculating an approximate diversity score is to match the first two moments (mean and standard deviation) of the return distribution associated with the actual pool:

Assume the actual collateral pool consists of n bonds, each with face value F_i and a default probability p_i that is implied by the rating and maturity of the bond. Let X_i be the nondefault indicator for bond i; $X_i = 0$, when bond i defaults and $X_i = 1$ otherwise. Thus $X_i = 0$ with probability p_i and X_i=1 with probability $q_i = 1 - p_i$. Assume also that the correlation coefficient of default between bond i and j is ρ_{ij}. After allowing for defaults, the remaining notional amount of the collateral pool is

(1) $P = \sum_{i=1}^{n} X_i F_i$.

Assume there exists a homogenous pool with D identical, uncorrelated bonds and that each identical bond has the same face value F, the same rating and the same maturity (which is the average life of the actual collateral pool). Let the default probability implied by the average rating and maturity be p and the indicator of bond i nondefault be Y_i; hence, Y_i=0, with probability p and $Y_i = 1$ with probability $q = 1 - p$. If P^* is the total notional of the homogenous pool, then

(2) $P^* = \sum_{i=1}^{D} Y_i F$.

We intend to use this homogenous pool of identical assets to approximate the default profile of the original collateral pool. Of course, the initial notional amount of two pools has to be the same so that

(3) $\sum_{i=1}^{n} F_i = DF$

In addition to (3), we match the first two moments of P and P^*. That is, the expected value of the notional amount of the two pools are equal

(4) $E(P) = E(P^*)$,

and the variance of the notional amount of the two pools are equal:

(5) $Var(P) = Var(P^*)$.

In order to match the first two moments, we observe that:

$X_i \sim \text{Binomial}(1, q_i)$, $corr(X_i, X_j) = \rho_{ij}$, $i = 1 \ldots n$ and

$Y_i \sim \text{Binomial}(1, q)$, $corr(Y_i, Y_j) = 0$, $i = 1, \ldots D$.

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[9] The material in this appendix was provided by Jeremy Gluck of Moody's Investors Sevices

Thus, $E(X_i) = q_i$, $Var(X_i) = p_i q_i$ and

$E(Y_i) = q$, $Var(Y_i) = pq$.

The first two moments of P are given by

(6) $E(P) = E(\sum_{i=1}^{n} X_i F_i) = \sum_{i=1}^{n} E(X_i) F_i = \sum_{i=1}^{n} q_i F_i$ and

(7)

$$Var(P) = Var(\sum_{i=i}^{n} X_i F_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sqrt{Var(X_i) Var(X_j)} F_i F_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sqrt{p_i q_i p_j q_j} F_i F_j$$

The first two moments of P^* are:

(8) $E(P^*) = F \sum_{i=1}^{D} E(Y_i) = Fq$

(9) $Var(P^*) = F^2 \sum_{i=1}^{D} Var(Y_i) = F^2 Dpq$

We can substitute (6) - (9) into (3) - (5) to obtain

(10) Average face value: $F = (\sum_{i=1}^{n} F_i) / D$

(11) Average default probability: $p = \dfrac{\sum_{i=1}^{n} p_i F_i}{\sum_{i=1}^{n} F_i}$

(12) Diversity Score: $D = \dfrac{(\sum_{i=1}^{n} p_i F_i)(\sum_{i=1}^{n} q_i F_i)}{\sum \sum \rho_{ij} \sqrt{p_i q_i p_j q_j} F_i F_j}$

Though we have not explicitly addressed the skewness of the return distribution, experimentation with a variety of portfolios suggests that the homogeneous portfolio consisting of D assets closely approximates the tail of the return distribution.

If the distribution of ratings of the collateral is not highly skewed, (12) can be simplified by replacing each of the default probabilities in (12) with the average default probability p that is defined in (11). In this case, (12) reduces to:

(13) Diversity Score: $$D = \frac{\sum_{i=1}^{n} F_i^{\,2}}{\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} F_i F_j}$$

Appendix C: Asset versus Default Correlations

The correlation between two discrete default events is defined as:

$$\rho_{ij} = \frac{PD_{ij} - PD_i PD_j}{\sqrt{PD_i(1-PD_i)PD_j(1-PD_j)}}$$

where PD_{ij} is the joint probability of the default of obligation *i* and *j*. In the context of the structural default model being applied to individual obligations, PD_{ij} is has to be equal to the probability of the value of the two assets both declining by more than their respective distances to default:

$$PD_{ij} = \Phi_2^{-1}\left(\Delta X_i < -DD_i, \Delta X_j < -DD_j, \rho_{ij}^V\right)$$

where $\Phi_2^{-1}(\ )$ denotes the inverse cumulative bivariate normal distribution and ρ_{ij}^V is the asset correlation. As shown in Table X, default correlations are lower than asset correlations, but they increase with default probabilities up to the 50% PD level and then decline symmetrically.[10]

Table X: Default Correlation v Asset Correlation			
	Asset Correlation		
PD_i=PD_j	**30%**	**20%**	**10%**
1%	4.61	2.41	0.93
5%	9.76	5.78	2.55
10%	12.91	8.00	3.71
20%	16.34	10.51	5.07
45%	19.32	12.76	6.35
50%	19.40	12.82	6.38
55%	19.32	12.76	6.35
80%	16.34	10.51	5.07
90%	12.91	7.99	3.71
95%	9.76	5.78	2.55
99%	4.60	2.41	0.93

The bivariate normal distribution is not a native Excel function, but free VBA code is available at http://my.dreamwiz.com/sjoo/source/bivariate_normal_distribution.txt)

Appendix C2: Simulating LGDs with the beta distribution

The beta distribution, which seems to be a popular way of modelling LGDs and recovery rates, has the following functional form:

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \text{ for } 0 \le x \le 1 \text{ with } B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

10 See Gersbach, and Lipponer (2003). For another in-depth examination of default correlation dynamics, see Zhou (2001).

where Γ is the gamma function. The α and β parameters define the distribution's "center" and "shape". If μ and σ are respectively the mean and standard deviation of the actual distribution, the center and shape parameters are defined as follows:

$$\alpha = \frac{\mu^2(1-\mu)}{\sigma^2} - \mu \quad \text{and} \quad \beta = \frac{\alpha}{\mu} - \alpha \;\Leftrightarrow\; \mu = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}$$

Excel does not have a beta distribution simulator, but Poptools SimTools (http://www.kellogg.nwu.edu/faculty/myerson/ftp/addins) do.