# On Bernstein-Heinz-Chern-Flanders inequalities

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*Abstract*

We give an interpretation of the Chern-Heinz inequalities for graphs in order to extend them to transversally oriented codimension one $C^2$-foliations of Riemannian manifolds. That contains Salavessa's work on mean curvature of graphs and fully generalize results of Barbosa-Kenmotsu-Oshikiri [**4**] and Barbosa-Gomes-Silveira [**3**] about foliations of 3-dimensional Riemannian manifolds by constant mean curvature surfaces. This point of view of the Chern-Heinz inequalities can be applied to prove a Haymann-Makai-Osserman inequality (lower bounds of the fundamental tones of bounded open subsets $\Omega \subset \mathbb{R}^2$ in terms of its inradius) for embedded tubular neighborhoods of simple curves of $\mathbb{R}^n$.

## 1. *Introduction*

Let $G(z)$ be the graph of a $C^2$-function $z = z(x, y)$ defined on $x^2 + y^2 < r^2$ with mean curvature $H(x, y)$ and Gaussian curvature $K(x, y)$. Heinz in [**17**] showed that, if $|H(x, y)| \geq b > 0$ then $b \leq 1/r$ and if $K(x, y) \geq b > 0$ then $b^{1/2} \leq 1/r$. As observed by Heinz, these inequalities are improvements of inequalities of the form $r \cdot \varrho(b) \leq 1$, for some constant $\varrho(b)$ depending on $b$, implicit in Bernstein's work [**5**], [**6**].

Chern in [**12**] and Flanders in [**14**], independently, extended these inequalities to graphs of $C^2$-functions $z = z(x_1, \ldots, x_n)$ defined on bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundaries $\partial\Omega$, showing that, if the mean curvature satisfies $|H(x_1, \ldots, x_n)| \geq b > 0$, then $n \cdot b \leq \mathrm{vol}_{n-1}(\partial\Omega)/\mathrm{vol}_n(\Omega)$, and if the scalar curvature $S(x_1, \ldots, x_n) \geq b > 0$, then $\sqrt{n(n-1)b} \leq \mathrm{vol}_{n-1}(\partial\Omega)/\mathrm{vol}_n(\Omega)$. These inequalities are known as the Chern-Heinz inequalities for graphs.

An immediate corollary of the Chern-Heinz inequalities is that a graph of a $C^2$-function $z : \mathbb{R}^n \to \mathbb{R}$ has constant mean curvature $H$ if and only if $H = 0$, and it has constant nonnegative scalar curvature $S \geq 0$ if and only if $S = 0$.

Isabel Salavessa in [**23**] extended the Chern-Heinz inequality to graphs $G(f)$ of smooth functions $f : M \to \mathbb{R}$. Showing that $n \cdot \inf_\Omega |H_{G(f)}| \leq \mathrm{vol}_{n-1}(\partial\Omega)/\mathrm{vol}_n(\Omega)$ for every oriented compact domain $\Omega \subset M$, where $n = \dim M$. In particular, if $M$ has Cheeger constant $h(M) = 0$ then the graph $G(f)$ is minimal. Recall that the Cheeger constant is given by $h(M) = \inf_\Omega(\mathrm{vol}_{n-1}(\partial\Omega)/\mathrm{vol}_n(\Omega))$, where the infimum is taken over all relatively compact subsets $\Omega$ of $M$ with smooth boundary, see [**10**].

We can look at a graph of a smooth function $z : \Omega \to \mathbb{R}$ as a leaf of a transversally oriented codimension one smooth foliation of $\Omega \times \mathbb{R}$ obtained by vertical translation of the graph. The Chern-Heinz inequalities says that for this particular foliation of $\Omega \times \mathbb{R}$, the infimum of the mean curvatures of these leaves are bounded above by the $h(\Omega)$. In fact, the infimum of the mean curvatures of the leaves of any transversally oriented codimension one $C^2$-foliation of $\Omega \times \mathbb{R}$ is bounded above by $h(\Omega)$.

In this paper, we show that the Chern-Heinz inequalities holds for any transversally oriented codimension one $C^2$-foliations of open subsets $\Omega$ of Riemannian manifolds $M$ in the sense that the infimum of the mean curvature of the leaves are bounded above by the fundamental tone $\lambda^*(\Omega)$ of $\Omega$. Here, the upper bound for the infimum of the mean curvatures of the leaves is given in terms of $\lambda^*(\Omega)$, although it is possible to express it in terms of the Cheeger constant $h(\Omega)$. Using this notation we state our main results.

Theorem 1·1. *Let $\mathcal{F}$ be a transversely oriented codimension one $C^2$-foliation of a connected open set $\Omega$ of an $(n+1)$-dimensional Riemannian manifold $M$. Then*

$$2\sqrt{\lambda^*(\Omega)} \geq n \cdot \inf_{F \in \mathcal{F}} \inf_{x \in F} |H^F(x)|$$

*where $H^F$ stands for the mean curvature function of the leaf $F$.*

Theorem 1·2. *Let $\mathcal{F}$ be a transversely oriented codimension one $C^2$ foliation of a complete Riemannian manifold $M$ with nonnegative Ricci curvature. Suppose that the leaves are complete hypersurfaces with the same constant mean curvature $H$. Then $H = 0$ and each leaf is stable. If furthermore a leaf $F$ is compact then it is totally geodesic and the Ricci curvature of $M$ in the direction normal to $F$ is zero.*

## 2. *Proof of theorem* 1·1 *and its consequences*

Let us recall that the fundamental tone $\lambda^*(\Omega)$ of an open subset $\Omega \subset M$ of a smooth Riemannian manifold $M$ is given by

$$\lambda^*(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}, \, f \in H_0^1(\Omega) \setminus \{0\} \right\}, \tag{2·1}$$

where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\varphi\|_\Omega^2 = \int_\Omega \varphi^2 + \int_\Omega |\nabla \varphi|^2$. We recall that Bessa and Montenegro [**8**] proved that for an open set $\Omega \subset M$ one has

$$\lambda^*(\Omega) \geq c(\Omega)^2/4, \tag{2·2}$$

for $c(\Omega) = \sup\limits_{X \in \mathcal{X}(\Omega)} \left( \inf\limits_\Omega \dfrac{\mathrm{div} X}{\|X\|_\infty} \right)$, where $\mathcal{X}(\Omega)$ is the set of all smooth vector fields in $\Omega$ such that $\|X\|_\infty = \sup_x |X(x)| < \infty$.

To prove theorem 1·1 we proceed as follows. Let $\Omega$ be an open set of a Riemannian manifold M and $\mathcal{F}$ a codimension one transversally oriented $C^2$-foliation. This just means we may choose a continuous unit vector field $\eta$ on $M$ that is normal to the leaves of $\mathcal{F}$. Represent by $H^F(x)$ the value of the mean curvature of the leaf $F$ at $x$ computed with respect to $\eta$. Set $b = \inf_{F \in \mathcal{F}} \inf_{x \in F} |H^F(x)|$. If $b = 0$ there is nothing to prove. Assume then, that $b > 0$. This implies that $H^F$ does not change sign. Hence, we may choose the unit vector field $\eta$ in such way that $H^F(x) > 0$ for any $x \in \Omega$. It is well known that $\mathrm{div}\,\eta = n \cdot H^F$. Therefore, $\inf_\Omega \mathrm{div}\,\eta = n \cdot \inf_\Omega H_F(x) \geq n \cdot b$. Since $\|\eta\|_\infty = 1$ and applying (2·2) to estimate $\lambda^*(\Omega)$ we obtain

$$2\sqrt{\lambda^*(\Omega)} \geq c(\Omega) \geq \inf_{\Omega} \frac{\operatorname{div}\eta}{\|\eta\|_\infty} = n \cdot \inf_{x\in\Omega} H_F(x) \geq n \cdot b$$

This proves Theorem 1·1.

This theorem has a number of interesting consequences, stated below as corollaries. It imposes strong restrictions for the existence of foliations by constant mean curvature hypersurfaces on open sets with zero fundamental tone or with Ricci curvature bounded below, see Corollaries 2·1 and 2·2.

COROLLARY 2·1. *Let $\mathcal{F}$ be a transversely oriented codimension one $C^2$-foliation of a Riemannian manifold $M$ for which $\lambda^*(M) = 0$. If the leaves of $\mathcal{F}$ have the same constant mean curvature then they are minimal submanifolds of $M$.*

REMARK. *The class of Riemannian manifolds $M$ with $\lambda^*(M) = 0$ is quite large. Beside all closed Riemannian manifolds, it contains all open Riemannian manifolds with asymptotically nonnegative Ricci curvature, see* [**11**]. *Recall that an open Riemannian manifold $M$ has asymptotically nonnegative Ricci curvature if $Ric_M(x) \geq -\psi(dist_M(x_0, x))$, for a continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \psi(t) = 0$, $x_0 \in M$.*

In [**4**], Barbosa-Kenmotsu-Oshikiri considered a transversely oriented codimension-one $C^2$-foliation $\mathcal{F}$ of an open subset $\Omega$ of an $n$-dimensional complete Riemannian manifold $M$ with Ricci curvature satisfying $Ric_M \geq (n-1)\kappa$. They assumed that the leaves were complete oriented hypersurfaces with the same constant mean curvature $H$. In their Theorem 3.1 they showed that, if $M$ is compact and $\kappa = 0$ then the leaves of $\mathcal{F}$ are totally geodesic and the Ricci curvature of $M$ is zero in the directions normal to the leaves. The remark above, the corollary 2·1 coupled with Theorem 2·4 implies as a particular case, the Barbosa-Kenmotsu-Oshikiri main result, Theorem 3.1.

In Theorem 3.8 of [**4**], they investigated the case when $M$ is the simply connected space form of constant sectional curvature $\kappa$. When $\kappa \leq 0$ and $H \geq (n-1)\sqrt{-\kappa}$ they concluded that $H = (n-1)\sqrt{-\kappa}$. The following corollary of Theorem 1·1 extends Theorem 3.8 of Barbosa-Kenmotsu-Oshikiri to $n$-dimensional Riemannian manifolds with Ricci curvature satisfying $\mathrm{Ric_M} \geq (n-1)k$.

COROLLARY 2·2. *Let $\mathcal{F}$ be transversely oriented codimension-one $C^2$-foliation of a complete $n$-dimensional Riemannian manifold $M$ with Ricci curvature $\mathrm{Ric_M} \geq (n-1)k$. Then*

i) $2\sqrt{\lambda^*(\mathbb{M}^n(k))} \geq \inf_{F\in\mathcal{F}} \inf_{x\in F} |H_F(x)|$, *where $\mathbb{M}^n$ represents the simply connected space form of constant sectional curvature $\kappa$ and dimension $n$.*

ii) *If $|H_F| \geq b > 0$ then $k = -a^2$ for some positive $a$ satisfying $(n-1)\cdot a \geq b$.*

Let $B_M(r)$ represent a geodesic ball of radius $r$ in the Riemannian Manifold $M$ whose Ricci curvature satisfies $\mathrm{Ric}_M \geq (n-1)\kappa$, and let $B_{\mathbb{M}(\kappa)}(r)$ represent the geodesic ball of radius $r$ in the simply connected $n$-dimensional space $\mathbb{M}(\kappa)$ of constant sectional curvature $\kappa$. By Cheng's Comparison Theorem (see [**11**]) we know that

$$\lambda^*(B_M(r)) \leq \lambda^*(B_{\mathbb{M}(\kappa)}(r)) \,.$$

Since the fundamental tone of $M$ can be also given by $\lambda^*(M) = \lim_{r\to\infty} \lambda^*(B_M(r))$, it follows that $\lambda^*(M) \leq \lambda^*(\mathbb{M}(\kappa))$. This and Theorem 1·1 gives us:

$$\inf_{F\in\mathcal{F}} \inf_{x\in F} |H_F(x)| \leq 2\sqrt{\lambda^*(M)} \leq 2\sqrt{\lambda^*(\mathbb{M}(\kappa))}$$

and so, $(i)$ is proved.

To obtain $(ii)$ observe that, if $|H_F| \geq b > 0$ then, from $(i)$ we have $\sqrt{\lambda^*(\mathbb{M}(\kappa))} \geq b/2$. But, $\lambda^*(\mathbb{M}(\kappa)) \neq 0$ if and only if $\kappa = -a^2$ in which case $\lambda^*(\mathbb{M}(\kappa)) = (n-1)^2a^2/4$. Thus the inequality $\sqrt{\lambda^*(\mathbb{M}(\kappa))} \geq b/2$ can occur only when $\kappa = -a^2$ and $(n-1)a \geq b$. This completes the proof of the Corollary.

Theorem 1·1 also has a version for scalar curvature provided the ambient manifold has nonpositive sectional curvature.

Corollary 2·3. *Let $M$ be an $(n+1)$-dimensional Riemannian manifold with nonpositive sectional curvature and let $\mathcal{F}$ be a transversely oriented codimension one $C^2$-foliation of a connected open set $\Omega \subset M$. Suppose that the scalar curvature $S_F$ of each leaf if nonnegative. Then*

$$\sqrt{\inf S} \leq 2\sqrt{\lambda^*(\Omega)}\ .$$

*In particular, if $\lambda^*(M) = 0$, $\Omega = M$, and all the leaves have the same has constant nonnegative scalar curvature $S > 0$, then $S = 0$.*

If $\inf S = 0$ there is nothing to prove. So, we assume that $\inf S = c > 0$. Let $p \in F$ and $\{e_1, \cdots e_n\}$ be an orthonormal basis for the tangent space $T_pF$ of the leaf $F \in \mathcal{F}$. The Gauss equation for the plane generated by $e_i, e_j$ is:

$$\tilde{K}(e_i, e_j) = \overline{K}(e_i, e_j) + \langle B(e_i, e_i),\, B(e_j, e_j)\rangle - |B(e_i, e_j)|^2$$

where $\tilde{K}$ represents the Gaussian curvature of $F$ and $\overline{K}$ is the one of the ambient space $\Omega \times \mathbb{R}$. Adding these equations results:

$$S(p) = \sum_{i,j} \overline{K}(e_i, e_j) + n^2\, H^2 - \|B\|^2$$

Since the sectional curvatures $K$ of $M$ are, by hypothesis, nonpositive, then $\overline{K} \leq 0$. It follows that

$$S(p) \leq n^2\, H^2\ .$$

and, since $S \geq c > 0$ then $H \geq \sqrt{c}/n > 0$. By Theorem 1·1 we have that

$$2\sqrt{\lambda^*(\Omega)} \geq n \inf_{F\in\mathcal{F}} \inf_{x\in F} |H_f(x)| \geq \sqrt{c} = \sqrt{\inf S}$$

This proves Corollary 2·3.

We turn to higher order mean curvatures. Recall that if $\psi : N \to M$ is an $n$-dimensional oriented hypersurface of $M$ and $k_1, \ldots, k_n$ are the principal curvatures at $p \in N$, then the $r$-th mean curvatures $H_r$ of $\psi(N)$ at $\psi(p)$ are defined by the identity

$$(1+tk_1)(1+tk_2)\cdots(1+tk_n) = 1 + \binom{n}{1} H_1\, t + \binom{n}{2} H_2\, t^2 + \cdots \binom{n}{n} H_n\, t^n$$

for all real number $t$. Thus, $H_1$ is the mean curvature of $\psi$, $H_n$ is the Gauss-Kronecker curvature. Since we always have $H_1^2 \geq H_2$, the following version of Theorem (1·1) for the 2-nd mean curvature $H_2$ is direct.

Corollary 2·4. *Let $\mathcal{F}$ be a transversely oriented codimension one $C^2$-foliation of a connected open set $\Omega$ of a Riemannian manifold $M$. Suppose that the leaves have the second mean curvature $H_2 \geq 0$. Then*

$$2\sqrt{\lambda^*(\Omega)} \geq n \cdot \inf_{F \in \mathcal{F}} \inf_{x \in F} (H_2^F)^{1/2}(x)$$

*where $H_2^F$ stands for the second mean curvature function of the leaf $F$. In particular, if $\lambda^*(M) = 0$, $\Omega = M$, and all the leaves have the same constant 2-nd mean curvature $H_2 \geq 0$ then $H_2 = 0$.*

If the principal curvatures at a point $q \in \psi(N)$ are positive, $k_i(q) > 0$ and if $H_r \geq 0$ then Gårding inequalities, $H_k^{1/k} \leq H_1$, $H_k^{(k-1)/k} \leq H_{k-1}$, $k = 1, \dots r$ holds everywhere, see [**18**], [**20**]. It was observed in proof of Theorem 1.3 of [**15**] that Gårding inequalities is still true under the weaker hypothesis $k_i(q) \geq 0$, if $H_r > 0$. The statement of the next corollary could be about foliations but we preferred to make it about graphs. It generalizes Corollary 1.7 of [**15**].

Corollary 2·5. *Let $M$ be a complete Riemannian manifold with $\lambda^*(M) = 0$ and $z : M \to \mathbb{R}$ be a $C^2$-function whose graph $G(z) \subset M \times \mathbb{R}$ has constant nonnegative $r$-th mean curvature $H_r \geq 0$. If $r \geq 3$ suppose that there is a point $q \in G(z)$ with all principal curvature $k_i(q) \geq 0$. Then $H_r = 0$.*

The cases $r = 1$ and $r = 2$ are covered by Theorem 1·1 and Corollary 2·1. Now, suppose that $r \geq 3$ and the graph has a point $q$ with principal curvatures $k_i(q) \geq 0$. If $H_r \geq 0$ is constant then we may suppose that $H_r > 0$, otherwise there is nothing to prove. Then we have that $H_1 \geq H_r^{1/r} > 0$. Since $\lambda^*(M) = 0$, by Theorem 1·1, $0 = (2/n)\sqrt{\lambda^*(M)} \geq \inf_M H_1 \geq H_r^{1/r} > 0$. A contradiction.

## 3. *Proof of Theorem* 1·2

Let $M$ be a complete Riemannian manifold with Ricci curvature satisfying $\mathrm{Ric}_M \geq 0$. Then we have

$$\lambda^*(M) = 0\,. \tag{3·1}$$

Let $\mathcal{F}$ be transversely oriented codimension-one $C^2$-foliation of $M$. Then, there exists a unit vector field $\eta$ in $M$ normal to all the leaves of $\mathcal{F}$. Let $H_F$ represent the mean curvature of the leaf $F$ computed with respect to $\eta$. Assume the leaves of $\mathcal{F}$ are complete hypersurfaces with the same constant mean curvature $H$, that is, $H_F = H$, where $H$ is a constant. It then follows from (3·1) and Theorem 1·1 that each leaf $F \in \mathcal{F}$ is minimal.

Fix a leaf $F$. On $F$ consider the Jacobi operator:

$$L = \triangle + \mathrm{Ric}\,(\eta) + |A|^2 \tag{3·2}$$

where $\triangle$ represents the Laplace operator in $F$, $A$ represents the second fundamental form of $F$ computed with respect to $\eta$, and $\mathrm{Ric}\,(\eta)$ is the Ricci curvature of $M$ in the direction of $\eta$.

Let $C$ be a compact set of $F$ with smooth boundary $\partial C$. Take another compact set $C'$ such that $C \subset C'$. We may assume that the only solution of the problem

$$\begin{aligned} L(v) &= 0 \qquad \text{in } C' \\ v &= 0 \qquad \text{in } \partial C' \end{aligned}$$

is the function $v \equiv 0$. This means that zero is not an eigenvalue of the operator $L$ in $C'$. Thus there exists a positive function $u \in C^{\infty}(C')$ with

$$\begin{aligned} L(u) &= 1 \qquad \text{in } C' \\ u &= 0 \qquad \text{in } \partial C' \,. \end{aligned}$$

Consider now the variation $\psi_t : C' \to M$ of $C'$ given by $\psi_t(x) = \exp_x(t\,u(x)\eta(x))$ for $|t| < \epsilon$. Represent by $H(t)$ the mean curvature of the immersion $\psi_t$. It has derivative at $t = 0$ given by $2H'(0) = L(u) = 1$ on $C'$ (see [**22**]). Since $H(0) = 0$, then, for small $0 < t < \epsilon$, $H(t) > 0$. Now we analyze some possibilities for the behavior of $u$.

If $u$ is positive at some interior point of $C'$ then $\psi_t(C')$ must have a tangency point with some leaf of $\mathcal{F}$, and this is not possible by the maximum principle. Thus $u \leq 0$.

If $u \leq 0$ and it is zero at an interior point of $C'$ we have that $u \equiv 0$ and this is impossible since $Lu = 1$. Thus $u < 0$ in the interior of $C'$ and $u = 0$ on $\partial C'$. Setting $w = -u$ we have that $w > 0$ in the interior of $C'$ with $L(w) < 0$.

Let $u_1$ be the first eigenfunction of $C$, i.e,

$$\begin{aligned} L(u_1) + \lambda_1(C)u_1 &= 0 \qquad \text{on } C \\ u_1 &= 0 \qquad \text{on } \partial C \end{aligned}$$

Furthermore we know that $u_1$ is not zero in the interior of $C$. Hence, we may assume it is positive in the interior of $C$.

Suppose by contradiction that $\lambda_1(C) < 0$. Define in $C$ the function $h = w - t\,u_1$ This is possible since $C \subset C'$. Observe that on $\partial C$ we have $h = w > 0$ and independent of $t$. Hence, we may choose $t$ such that $h \geq 0$ and is zero at some point in the interior of $C$. Observe that

$$L(h) = L(w) - tL(u_1) \leq t\lambda_1(C)u_1 < 0$$

on $C'$. Then $\triangle h < 0$ and $h$ has a minimum in the interior of $C$. By the maximum principle $h$ is constant. Since h $= 0$ at one point, then it is everywhere zero. But this is a contradiction since, as we have seen, $h > 0$ at $\partial C$.

This contradiction shows that $\lambda_1(C) \geq 0$. Suppose that $F$ is not stable. Then there exists a function $f : F \to \mathbb{R}$ such that $\int_F f = 0$ and $L(f) + \bar{\lambda}_1 f = 0$ with $\bar{\lambda}_1 < 0$. Here $\bar{\lambda}_1$ is the first eigenvalue of the twisted problem:

$$\begin{aligned} L(f) + \bar{\lambda} f &= 0 \qquad \text{on } F \\ \int_F f &= 0 \end{aligned}$$

But then $\bar{\lambda}_1 < 0$ would be the first eigenvalue $\lambda_1$ of $L$ on the set $\{x \in F;\ f(x) > 0\}$. But, exhausting this set with a family of compact sets we may show that $\lambda_1 \geq 0$. This contradiction shows that $F$ is stable as we wished to prove.

Assume now that $F$ is compact. Since $F$ is stable, by a result of Fisher-Colbrie [**13**] there is a positive function $g : F \to \mathbb{R}$ solution to

$$\triangle g + (\mathrm{Ric}\,(\eta) + |A|^2)g = 0\,.$$

Integrating this equation over $F$ we obtain

$$0 = \int_F \triangle g + (\mathrm{Ric}\,(\eta) + |A|^2)g = \int_F (\mathrm{Ric}\,(\eta) + |A|^2)g$$

Since $\mathrm{Ric}\,(\eta) \geq 0$ and $g > 0$ then we conclude that

$$\mathrm{Ric}\,(\eta) = 0$$

and

$$|A|^2 = 0$$

Consequently $F$ is totally geodesic and the Ricci curvature of $M$ in direction normal to $F$ is zero. This concludes the proof of Theorem 1·2.

REMARK. *For $n = 3$ Schoen* [**24**] *showed that a complete stable minimal surface in a 3-dimensional Riemannian manifold with nonnegative Ricci curvature is totally geodesic. Thus, for $n = 3$, Theorem 1·2 fully generalizes Theorem 3.1 of* [**4**].

## 4. *Haymann-Makai-Osserman inequality*

Recall that the inradius $\rho(\Omega)$ of a connected open set $\Omega$ of a Riemannian manifold $M$ is defined as $\rho(\Omega) = \sup\{r > 0;\, B_M(r) \subset \Omega\}$, where $B_M(r)$ represents a geodesic ball of radius $r$ of $M$. In [**19**], Makai proved that the fundamental tone $\lambda^*(\Omega)$ of a simply connected bounded domain $\Omega \subset \mathbb{R}^2$ with inradius $\rho$ and smooth boundary is bounded below by $\lambda_1(\Omega) \geq 1/4\rho^2$. Unaware of Makai's result, Haymann [**16**] proved years later that $\lambda_1(\Omega) \geq 1/900\rho^2$. Osserman [**21**] among other things improved Haymann's estimate back to $\lambda_1(\Omega) \geq 1/4\rho^2$. Recently, Haymann-Makai-Osserman inequality was improved by Bañelos-Carroll in [**2**] to $0.6197/\rho^2$.

Our next result extends of Haymann-Makai-Osserman inequality to embedded tubular neighborhoods of simple smooth curves in $\mathbb{R}^n$ with variable radius.

THEOREM 4·1. *Let $\gamma : I = (\alpha, \beta) \subset \mathbb{R} \to \mathbb{R}^n$ be a simple smooth curve and $T_\gamma(\rho(t))$ be an embedded tubular neighborhood of $\gamma$ with variable radius $\rho(t)$ and smooth boundary $\partial T_\gamma(\rho(t))$. Let $\rho_o = \sup_t \rho(t) > 0$ be its inradius. Then*

$$\lambda^*(T_\gamma(\rho(t))) \geq \frac{(n-1)^2}{4\rho_o^2} \tag{4·1}$$

Consider the family of balls of $\mathbb{R}^n$ $B_t = B_{\mathbb{R}^n}(\gamma(t), \rho_o)$, $t \in I$ with center at $\gamma(t)$ and radius $\rho_o$. The set $\{S_t = \partial B_t \cap T_\gamma(\rho(t))\}$ is a smooth codimension one transversally oriented foliation of $T_\gamma(\rho(t)) \setminus (B_\alpha \cup B_\beta)$. Pushing the family $B_t$ little further one can fill $T_\gamma(\rho(t))$ with a smooth codimension one transversally oriented foliation such that the mean curvature of the leaves is constant $1/\rho_o$. The Theorem 1·1 yields that $\lambda^*(T_\gamma(\rho)) \geq (n-1)^2/4\rho^2$.

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