Prescribed m-curvature spheres in conformally flat manifolds

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UFC

Setup

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OBS: Euclidean spheres are umbilic as submanifolds of $\bar{M}$.

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The second fundamental form of M is given by $\bar{A} = (\bar{a}_{ij})$ where

$$\bar{\nabla}_{E_i} N = \sum \bar{a}_{ij} E_j \ .$$

The m-curvatures

The basic invariants associated to $\bar{A}$ are the m-curvatures S_m of M, $0 < m \leq n$, given by:

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When $\bar{A}$ is diagonal, that is, $\bar{a}_{ij} = k_i \delta_{ij}$ then

$$S_m = \sum_{1 \le i_1 < \ldots < i_m \le n} k_{i_1} \ldots k_{i_m}$$

The Problem

Given a function $\psi : \mathcal{D} \to \mathbb{R}$, does exist a function $z : S^n(1) \to \mathbb{R}$, $z > 0$, such that it satisfies the equation

$$(*) \quad \bar{S}_m(z(u)u) = \psi(z(u)u) \quad \text{for all} \quad u \in S^n(1)$$

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When $\psi =$ const., the problem is trivial.

The Main Hypothesis

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If $k(r)$ is positive, the contact point p is elliptical.

In this case $S_m(p) \geq \binom{n}{m} k(r)^m$

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a) $\psi > 0$

b) $\psi(u, r) > \binom{n}{m} k(r)^m$ for $r \leq r_1$ and
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If we further assume that the curvature of $\bar{M}$ *is constant and nonnegative than we can also exhibit an a priori* C^2 *bound for the solution.*

Hence, if the curvature of $\bar{M}$ *is constant and nonnegative, and if we have the validity of the above hypothesis then there exists a unique solution of* $(*)$.

Continuity method

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Consider the set $T = \{t \in [0,1];\ S_m(t,u,z(u)) = \Psi(t,u,z(u))$ has solution$\}$ and show it is nonempty, open and closed.

Applying continuity method

For $t \in [0,1]$ define

$$\Psi(t,u,r) = t\psi(u,r) + (1-t)f(r)\binom{n}{m}k(r)^m$$

where $f : (0,a) \to \mathbb{R}$, $\quad a > r_2 \quad$ satisfies

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Lemma: *the function Ψ satisfies the properties listed as the hypothesis about ψ.*

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Proof of the lemma

Since $\Psi(t,u,r) = t\psi(u,r) + (1-t)f(r)\binom{n}{m}k(r)^m$ it is clear that:

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Take $f(t) = e^{-a^2t+b}$ choosing:
$a^2 = m \max_{r_1 \leq r \leq r_2}(k' + k^2/\rho)$ and
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Hence, the result is proved, that is, for each t the function Ψ satisfies the same set of hypothesis as ψ.

The continuity method

For $t = 0$ and r such that $f(r) = 1$, we have $\Psi = \binom{n}{m} k(r)^m$. Hence $z = r$ is solution of the equation $S_m = \Psi$.

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The closeness of T depends on the existence of C^0, C^1 and C^2 a priori estimates.

C^0 estimate

Assume there exists a solution $z(u)$ of the equation $(*)$. I claim that $r_1 \leq z(u) \leq r_2$ for each $u \in S^n(1)$.

C^0 estimate

PROOF: Assume $\max(z(u)) = r > r_2$. Let $z(u_0) = r$. Then $M \subset \bar{B}_r(0)$ and $\partial \bar{B}_r(0)$ has a point of contact (u_0, r) with M.

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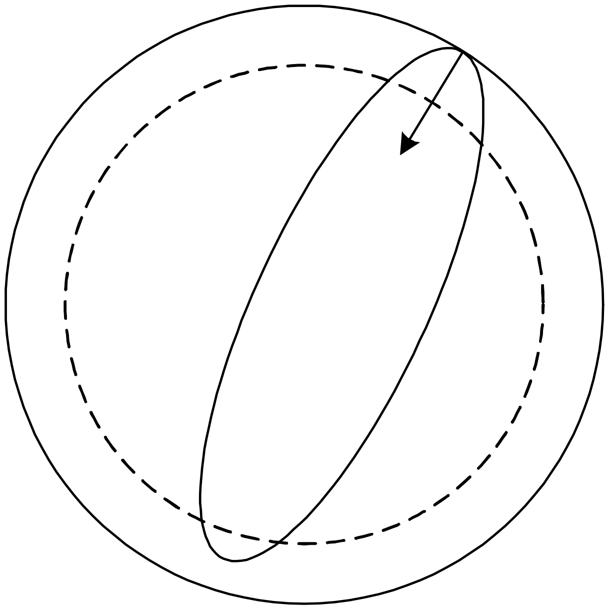

Since $\bar{S}_m = \psi < \binom{n}{m} k(r)^m$ by maximum principle we get a contradiction.

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If $f(u_0) = 0$ then $\text{grad}\, z(u) \equiv 0$. Hence, bounded.

Assume $f(u_0) > 0$. The function $\ln f(u) = \ln z_1 + Az(u)$ is well defined in a neighborhood $\mathcal{U}$ of u_0 and also has a maximum at u_0.

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$z_{ij} = 0$ for $i \neq j$ and

$$\begin{aligned}
\bar{a}_{11} &= \frac{\rho}{W^3}(-zz_{11} + 2z_1^2 + z^2) - \frac{z\rho'}{W} \\
\bar{a}_{ii} &= \frac{\rho}{zW}(-z_{ii} + z) - \frac{z\rho'}{W} \quad \text{for} \quad i > 1
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and furthermore $z_{111} \leq 2A^2 z_1^3$ and

$z_{ii1} + \frac{z_{ii}^2}{z_1} + Az_{ii}z_1 + z_1 \leq 0$ for $i > 1$.

Take the derivative of the equation $\bar{S}_m = \psi$ with respect to e_1 at the point u_0, to obtain:

$$\sum \frac{\partial \bar{S}_m}{\partial \bar{a}_{ij}} \frac{\partial \bar{a}_{ij}}{\partial z_{kl}} z_{kl,1} + \sum \frac{\partial \bar{S}_m}{\partial \bar{a}_{ij}} \frac{\partial \bar{a}_{ij}}{\partial z_k} z_{k,1} + \sum \frac{\partial \bar{S}_m}{\partial \bar{a}_{ij}} \frac{\partial \bar{a}_{ij}}{\partial z} z_1 = \psi_z z_1$$

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Using the particular expressions of the $\bar{a}_{ij}$ at the point u_0 one reduces this to:

$$\sum \frac{\partial \bar{S}_m}{\partial \bar{a}_{ii}} \frac{\partial \bar{a}_{ii}}{\partial z_{ii}} z_{ii,1} + \sum \frac{\partial \bar{S}_m}{\partial \bar{a}_{ii}} \frac{\partial \bar{a}_{ii}}{\partial z_1} z_{11} + \sum \frac{\partial \bar{S}_m}{\partial \bar{a}_{ii}} \frac{\partial \bar{a}_{ii}}{\partial z} z_1 = \psi_z z_1$$

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Now one proceeds using in equation all the information known about the a_{ii}, z_{ii} and $z_{ii,1}$ and the fact that

$$\sum \frac{\partial \bar{S}_m}{\partial \bar{a}_{ii}} \bar{a}_{ii} = m\bar{S}_m = m\psi$$

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to produce an inequality of the type:
$\alpha A^2 + \beta A + \gamma \leq 0$

where $\alpha = \frac{z z_1^2}{\lambda W^5}(z_1^2 - z^2)\frac{\partial \bar{S}_m}{\partial \bar{a}_{11}}$

Since $\alpha A^2 + \beta A + \gamma \leq 0$ must be valid for all A, then $\alpha < 0$.

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$$\alpha = \frac{z z_1^2}{\lambda W^5}(z_1^2 - z^2)\frac{\partial \bar{S}_m}{\partial \bar{a}_{11}} \ ,$$

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Hence

$$|\mathrm{grad}\, z| = z_1 < r_2$$