1 Finance

1.1 Interest

Often I find quotes from financial firms concerning amounts of interest made for a certain year. I believe these business types call this the *APR*, or the Annual Percentage Rate. As useful as this may seem, it can be slightly confusing if you learned all about interest calculation from a mathematical point of view. For example, from a mathematical perspective, we are taught to calculate interest as follows

$$M(t) = M_0 e^{\lambda t}$$

where $M(t)$ is the amount of money per unit time, M_0 is the initial amount of money invested, λ is the *instantaneous* interest rate, and t is time. I call λ the instantaneous interest rate because it can only be used when calculateing interest with continuous compounding. Thus it is inherently different than the APR. In fact, business types (hereafter referred to as suits) *never* use the instantaneous rate. Their APR is essentially the percentage increase in an investment in one years time. For example, if the instantaneous interest (λ) rate is 5% then a \$100 investment in one years time will grow to \$105.13 (rounded to nearest cent). Thus the APR is %5.13. What is the relation between the two you ask? Well, if we represent APR as λ' then the above equation becomes

$$M(t) = M_0 \left(1 + \lambda'\right)^t$$

Since these two equations must produce the same output, we can set them equal to each other.

$$M_0 e^{\lambda t} = M_0 \left(1 + \lambda'\right)^t$$

$$\lambda = \ln\left(1 + \lambda'\right)$$

Here are some values of random APRs and the corresponding continuous interest rate.

APR	λ'
5%	4.879%
10%	9.531%
15%	13.976%
20%	18.232%
25%	22.314%
30%	26.236%
40%	33.647%
50%	40.547%
60%	47.0%
70%	53.063%

Let's look at some results from the kind of compunding at different interest rates, shall we? The table below represents the growth of a \$1,000 investment with different amounts of interest.

Year	5%	10%	15%	20%
1	$1,050.00	$1,100.00	$1,150.00	$1,200.00
4	$1,215.51	$1,464.10	$1,749.01	$2,073.60
7	$1,407.10	$1,948.72	$2,660.02	$3,583.18
10	$1,628.89	$2,593.74	$4,045.56	$6,191.74
13	$1,885.65	$3,452.27	$6,152.79	$10,699.32
16	$2,182.87	$4,594.97	$9,357.62	$18,488.43
19	$2,526.95	$6,115.91	$14,231.77	$31,948.00
22	$2,925.26	$8,140.27	$21,644.75	$55,206.14
25	$3,386.35	$10,834.71	$32,918.95	$95,396.22
28	$3,920.13	$14,420.99	$50,065.61	$164,844.66
31	$4,538.04	$19,194.34	$76,143.54	$284,851.58

1.2 Continual money adding or withdrawing

If an invester continuously adds money to his nest egg than the above equation is not valid. We must start by solving the first order ODE

$$\frac{dM}{dt} = \lambda M + k$$

Here k is the rate at which money is added or withdrawn. A positive k represents money added and a negative k is used for money withdraws. The solution to this ODE is

$$M(t) = M_0 e^{\lambda t} + \left(\frac{k}{\lambda}\right)\left(e^{\lambda t} - 1\right)$$

The first term in this equation is our first equation while the second takes care of the money we are adding. Let's see what happens if we start with the the same $1,000 as above but add another $1,000 each year.

Year	5%	10%	15%	20%
1	$2,074.80	$2,149.21	$2,223.25	$2,296.96
4	$5,632.51	$6,333.46	$7,108.16	$7,962.10
7	$9,751.00	$11,902.71	$14,537.50	$17,751.45
10	$14,518.68	$19,315.38	$25,836.60	$34,667.45
13	$20,037.86	$29,181.64	$43,021.10	$63,898.30
16	$26,426.99	$42,313.64	$69,156.60	$114,409.21
19	$33,823.22	$59,792.33	$108,905.41	$201,692.05
22	$42,385.28	$83,056.46	$169,358.39	$352,516.81
25	$52,296.93	$114,021.02	$261,299.81	$613,142.00
28	$63,770.90	$155,234.85	$401,131.23	$1,063,502.32
31	$77,053.46	$210,090.45	$613,797.33	$1,841,724.95