A Discrete Scheme of Landweber Iteration for Solving Nonlinear Ill-Posed Problems

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In this paper we consider the finite dimensional approximation of Landweber iteration for nonlinear ill-posed problems and propose an a posteriori stopping rule to choose the termination index of the iteration. Under certain conditions, we obtain convergence, a pseudo-optimality estimate, and rates of convergence. A numerical example is given to test the theoretical assertions.

1. INTRODUCTION

With the growing interest in the applied sciences, a lot of attention has been paid to the study of nonlinear ill-posed problems, that is, problems which can be formulated as a nonlinear operator equation

\[ F(x) = y, \]  

where \( F \) is a continuous and Fréchet differentiable nonlinear operator with domain \( D(F) \) in the real Hilbert space \( X \) and with its range \( R(F) \) in the real Hilbert space \( Y \) and \( y \in R(F) \). In general, problem (1) is ill-posed.
in the sense that the solution of (1) does not depend continuously on the right hand side, which is often obtained by measurement and hence contains error. Let us assume that $y_\delta$ is an available approximation of $y$ satisfying

$$\|y_\delta - y\| \leq \delta,$$  

where $\delta > 0$ is a given noise level. Then the computation of a stable approximate solution of (1) from $y_\delta$ becomes an important topic.

Tikhonov regularization is the most well known method for solving nonlinear ill-posed problems, and it has received a lot of attention in recent years (cf. [2, 5, 6, 14, 15]). Iteration methods are also attractive since they are straightforward to implement for the numerical solution of nonlinear ill-posed problems. In [4] Landweber iteration [7] was extended to the study of nonlinear problems. Its basic idea is the following: let $x_0 \in D(F)$ be an initial guess of the sought solution of (1), let $x_0^\delta := x_0$, and define $x_k^\delta$ successively by

$$x_k^\delta = x_{k-1}^\delta - F'(x_{k-1}^\delta)^* (F(x_{k-1}^\delta) - y_\delta), \quad k = 1, 2, \ldots;$$

then use $x_k^\delta$ to approximate the solution of (1). Here $F'(x)$ denotes the Fréchet derivative of $F$ at $x \in D(F)$, and $F'(x)^*$ denotes the adjoint of $F'(x)$. By imposing some conditions on the nonlinearity of $F$, convergence and rates of convergence can be obtained provided that the termination index of iteration is chosen according to a suitable stopping rule. In [4] the generalized discrepancy principle has been investigated where the stopping index $k_\delta$ is determined by

$$\|F(x_k^\delta) - y_\delta\| > \tau \delta \geq \|F(x_{k_\delta}^\delta) - y_\delta\|, \quad 0 \leq k < k_\delta,$$

with $\tau$ being a suitable large positive number. However, all the results in [4] are obtained in the infinite dimensional setting. For numerical realization of Landweber iteration, we should consider finite dimensional approximations. Recently, based on nonlinear Landweber iteration Scherzer [13] presented a multi-level algorithm in which at each level a stopping rule was required to terminate the iteration and several such rules were suggested. However, some difficulties will arise when this method is applied to practical problems since his stopping rules depend heavily on the knowledge of the solution.

In this paper, we contribute to the study of finite dimensional approximation of Landweber iteration. The main topic is a suitable stopping rule for choosing the termination index of iteration. The paper is organized as follows. In Section 2 we give the description of the finite dimensional
approximation of Landweber iteration and present a stopping rule based on a modification of (4), then we prove the convergence. Section 3 contributes to an estimate of the approximate solution, and in Section 4 we point out that rates of convergence can be derived if the sought solution admits some smoothness conditions. Finally in Section 5 a numerical example is performed to verify theoretical results. We conjecture that our stopping rule combined with the skills in [13] may produce an effective multilevel algorithm; we leave this for future research.

2. STOPPING RULE AND CONVERGENCE

For numerical computations, we have to consider finite dimensional approximations of Landweber iteration. Therefore it is necessary to introduce a sequence of subspaces to approximate $X$. We assume that $(X_n)$ is a sequence of finite-dimensional subspaces of $X$ with the property

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} X_n = X. \quad (5)$$

We also assume that the interior of $D(F)$ is not empty and for all $n$

$$C_n := D(F) \cap X_n \neq \emptyset.$$

Now we can formulate the finite dimensional approximation of Landweber iteration into the following form.

Let $x_0^{\delta,n} := P_n x_0$ and define $(x_k^{\delta,n})$ successively by

$$x_k^{\delta,n} = x_{k-1}^{\delta,n} - P_n F' \left( x_{k-1}^{\delta,n} \right) \left( F( x_{k-1}^{\delta,n} ) - y_\delta \right), \quad k = 1, 2, \ldots, \quad (6)$$

where $P_n$ denotes the ortho-projector of $X$ onto $X_n$.

In this paper, instead of $x_k^\delta$ we shall use $x_k^{\delta,n}$ to approximate the exact solution of (1). To study the approximation property of $x_k^{\delta,n}$, it is assumed that the nonlinear operator $F$ satisfies the following three local properties in an open ball $B_\rho(x_0)$ of radius $\rho$ around $x_0$.

Assumption 2.1. (i) Problem (1) is properly scaled, i.e., $F$ is Fréchet differentiable in $B_\rho(x_0)$, and the Fréchet derivative $F'(x)$ at $x \in B_\rho(x_0)$ satisfies

$$\| F'(x) \| \leq 1, \quad x \in B_\rho(x_0) \subset D(F); \quad (7)$$

(ii) There is a constant $\eta < \frac{1}{2}$ such that for every pair $x, z \in B_\rho(x_0)$ there holds

$$\| F(x) - F(z) - F'(z)(x - z) \| \leq \eta \| F(x) - F(z) \|; \quad (8)$$
(iii) There exists a positive constant $C_0$ and a sequence of positive numbers $\{\gamma_n\}$ satisfying $\lim_{n \to \infty} \gamma_n = 0$ such that
\[
\|F'(x)(I - P_n)\| \leq C_0 \gamma_n, \quad x \in B_{\rho}(x_0).
\]

This assumption can be verified for many concrete problems. For the verification of (ii), one can find some examples in [4]. Moreover (iii) characterizes the approximation property of $X$ and one can consult [10] for the verification of (iii) for two parameter identification problems arising from differential equations.

To guarantee the convergence of $x_{k, n}^\delta$, the termination index of iteration should be chosen properly. In this paper, by modifying (4), we propose the following stopping rule.

Rule 2.1. Let $c_1, c_2 > 0$ and $\tau \geq 1$ be given constants, and let $k_0(\delta, n)$ be the largest integer satisfying $k_0(\delta, n) \leq (c_1 \delta + c_2 \gamma_n)^{-2}$.

(i) Choose $k(\delta, n)$ to be the first integer such that $k(\delta, n) \leq k_0(\delta, n)$ and
\[
\|F(x_{k(\delta, n)}^\delta) - y_\delta\| \leq \tau \delta; \tag{10}
\]

(ii) If there is no integer $k(\delta, n)$ satisfying $k(\delta, n) \leq k_0(\delta, n)$ such that (10) holds, then choose $k(\delta, n) = k_0(\delta, n)$.

Obviously, Rule 2.1 is well defined, and for the $k(\delta, n)$ defined by it, if $k(\delta, n) < k_0(\delta, n)$, then we always have
\[
\|F(x_{k(\delta, n)}^\delta) - y_\delta\| \leq \tau \delta; \tag{11}
\]
moreover, if $k(\delta, n) \geq 1$, then for all integers $0 \leq k < k(\delta, n)$ there holds
\[
\|F(x_k^\delta) - y_\delta\| > \tau \delta.
\]

In the following we shall consider the convergence of $x_{k(\delta, n)}^\delta$. The following lemma will be useful.

Lemma 2.1. Let Assumption 2.1 hold and let $x_\ast$ be an arbitrary solution of (1) in $B_{\rho/\delta}(x_0)$ and denote by $k(\delta, n)$ the termination index of iteration chosen by Rule 2.1. If $\tau > \frac{2(1 + n)}{1 + \frac{1}{2}\gamma_n}$, then $x_{k(\delta, n)}^\delta \in B_{\rho}(x_0)$ for all integers $k \leq k(\delta, n)$ provided that $n$ is sufficiently large. Moreover, for all integers $0 \leq l \leq k(\delta, n)$ there holds
\[
\|x_k^\delta - x_\ast\| \leq \|x_l^\delta - x_\ast\| + C_1 \|((I - P_n)x_\ast\|, \tag{12}
\]
where $C_1 = (C_0(1 + \eta)/c_2)^{1/2}(1/((1 - 2\eta)\tau - 2(1 + \eta)))$. 

Proof. When \( k(\delta, \eta) = 0 \), the assertion is trivial. Therefore in the following we assume \( k(\delta, \eta) \geq 1 \). We note that if \( x_j^{\delta, n} \in B_{\tau/4}(x_\ast) \subseteq B_\eta(x_\ast) \) for some \( j \), then the definition of \( x_j^{\delta, n} \) gives

\[
\|x_{j+1}^{\delta, n} - x_\ast\|^2 = \|x_j^{\delta, n} - x_\ast\|^2 + \|x_j^{\delta, n} - x_j^{\delta, n}\|^2 \\
+ 2\langle x_j^{\delta, n} - x_j^{\delta, n}, x_j^{\delta, n} - x_\ast \rangle \\
= \|x_j^{\delta, n} - x_\ast\|^2 + \|P_n F(x_j^{\delta, n}) - (F(x_j^{\delta, n}) - y_\delta)\|^2 \\
- 2\langle F'(x_j^{\delta, n}) (x_j^{\delta, n} - P_n x_\ast), (x_j^{\delta, n} - y_\delta) \rangle.
\]

(13)

By using Assumption 2.1(ii) one can obtain

\[
-\langle F'(x_j^{\delta, n}) (x_j^{\delta, n} - P_n x_\ast), (F(x_j^{\delta, n}) - y_\delta) \rangle \\
\leq \eta \left\| F(x_j^{\delta, n}) - F(P_n x_\ast) \right\| \left\| F(x_j^{\delta, n}) - y_\delta \right\| \\
- \langle F'(x_j^{\delta, n}) (x_j^{\delta, n} - P_n x_\ast), (F(x_j^{\delta, n}) - y_\delta) \rangle \\
\leq -(1 - \eta) \left\| F(x_j^{\delta, n}) - y_\delta \right\|^2 \\
+ (1 + \eta) \left\| F(P_n x_\ast) - y_\delta \right\| \left\| F(x_j^{\delta, n}) - y_\delta \right\|.
\]

(14)

Since Assumption 2.1(i) implies \( \|P_n F(x_j^{\delta, n}) - F(x_j^{\delta, n}) - y_\delta\| \leq \|F(x_j^{\delta, n}) - y_\delta\| \), therefore by substituting (14) into (13) and noting that

\[
\left\| F(P_n x_\ast) - y_\delta \right\| \leq \delta + \left\| F(P_n x_\ast) - F(x_\ast) \right\|
\]

it yields

\[
\|x_{j+1}^{\delta, n} - x_\ast\|^2 \leq \|x_j^{\delta, n} - x_\ast\|^2 - (1 - 2\eta) \left\| F(x_j^{\delta, n}) - y_\delta \right\|^2 \\
+ 2(1 + \eta)(\delta + \left\| F(P_n x_\ast) - F(x_\ast) \right\|) \left\| F(x_j^{\delta, n}) - y_\delta \right\|.
\]

(15)

Now let \( 0 \leq j < k(\delta, \eta) \) be arbitrary and suppose that \( x_j^{\delta, n} \in B_{\tau/4}(x_\ast) \subseteq B_\eta(x_\ast) \). Since \( k(\delta, \eta) \geq 1 \), we always have \( \|F(x_j^{\delta, n}) - y_\delta\| > \tau \delta \). Therefore

\[
\|x_{j+1}^{\delta, n} - x_\ast\|^2 \leq \|x_j^{\delta, n} - x_\ast\|^2 - \frac{(1 - 2\eta) \tau - 2(1 + \eta)}{\tau} \left\| F(x_j^{\delta, n}) - y_\delta \right\|^2 \\
+ 2(1 + \eta) \left\| F(P_n x_\ast) - F(x_\ast) \right\| \left\| F(x_j^{\delta, n}) - y_\delta \right\| \\
\leq \|x_j^{\delta, n} - x_\ast\|^2 \\
+ \frac{(1 + \eta)^2 \tau}{(1 - 2\eta) \tau - 2(1 + \eta)} \left\| F(P_n x_\ast) - F(x_\ast) \right\|^2.
\]

(16)
By noting that \((I - P_n)^2 = I - P_n\) and
\[
F(P_n x_*) - F(x_*) = \int_0^1 F'(x_* + t(P_n - I) x_*)(P_n - I) x_* \, dt,
\]
from Assumption 2.1(iii) we have
\[
\|F(P_n x_*) - F(x_*)\| \leq C_0 \gamma_n\| (I - P_n) x_* \|. \tag{17}
\]
Inserting (17) into (16) it follows that
\[
\|x_{j+1}^{\delta,n} - x_*\|^2 \leq \|x_j^{\delta,n} - x_*\|^2 + C_0^2 \frac{(1 + \eta)^2 \tau}{(1 - 2\eta)\tau - 2(1 + \eta)} \gamma_n^2\| (I - P_n) x_* \|^2. \tag{18}
\]
Noting that \(x_0^{\delta,n} \in B_{\rho/2}(x_*)\) for sufficiently large \(n\) and \((j + 1)\gamma_n^2 \leq c_1^{-2}\), from (18) we deduce that \(x_j^{\delta,n} \in B_{3\rho/4}(x_*)\) for all \(0 \leq j \leq k(\delta, n)\) if \(n\) is large enough. Moreover, for any integers \(0 \leq l \leq k(\delta, n)\), we have
\[
\|x_k^{\delta,n} - x_*\|^2 \leq \|x_l^{\delta,n} - x_*\|^2 + C_0^2 \frac{(1 + \eta)^2 \tau}{c_1^2((1 - 2\eta)\tau - 2(1 + \eta))} \| (I - P_n) x_* \|^2.
\]
This proves assertion (12). \(\blacksquare\)

**Lemma 2.2.** Let Assumption 2.1(i) and (ii) hold, and let \(x_k\) be defined by (3) with \(y_\delta\) replaced by \(y\). If (1) is solvable in \(B_{\rho/4}(x_0)\), then \(x_k\) converges to a solution of \(x^* \in B_{\rho/4}(x_0)\) of (1).

**Proof.** See [4, Theorem 2.3]. \(\blacksquare\)

Now we can give the convergence results.

**Theorem 2.1.** Let Assumption 2.1 hold and suppose the nonlinear mapping \(x \mapsto F'(x)\) is continuous with respect to \(x \in B(x_0)\). If (1) is solvable in \(B_{\rho/4}(x_0)\), and \(k(\delta, n)\) is defined by Rule 2.1 with \(\tau > \frac{2(1 + \eta)}{1 - 2\eta}\), then
\[
x_k^{\delta,n} \rightarrow x^*, \quad \text{as } \delta \rightarrow 0, n \rightarrow \infty,
\]
where \(x^*\) is a solution of (1) in \(B_{\rho/4}(x_0)\) as defined in Lemma 2.2.

**Proof.** The result follows from Lemmas 2.1 and 2.2 by using the technique in [4, Theorem 2.4]. \(\blacksquare\)
3. AN ESTIMATE FOR THE APPROXIMATE SOLUTIONS

For ill-posed problems, the choice of the regularization parameter for continuous regularization or the stopping index for iteration methods is very important and has received considerable attention. The concept of pseudo-optimality of a parameter choice was first introduced in [8] for linear Tikhonov regularization, and then it was extended in [11] to the stopping rules of some iteration methods for solving linear ill-posed problems in Banach spaces. By considering these existing results, we hope to prove some similar results for Rule 2.1 when applied to Landweber iteration.

In this section we shall point out the possibility for Rule 2.1. Before doing this, let us give some preparatory results.

Lemma 3.1. Under Assumption 2.1, if \( x^* \) is a solution of (1) in \( B_{\rho/4}(x_0) \) and \( c_1 \) is chosen such that \( c_1 \geq 2(1 + \eta)/\rho \sqrt{1 - 2\eta} \), then for any integer \( k \leq k_0(\delta, n) \), we have \( x_k^{\delta,n} \in B_{3\rho/4}(x^*) \subset B_{\rho}(x_0) \) provided that \( n \) is large enough.

Proof. From (15) and (17) it is easy to show that

\[
\| x_{k+1}^{\delta,n} - x^* \|^2 - \| x_k^{\delta,n} - x^* \|^2 \leq \frac{(1 + \eta)^2}{1 - 2\eta} \left( \delta + C_0 \gamma_n \| (I - P_n)x^* \| \right)^2. \tag{19}
\]

Therefore, by noting that \( \sqrt{k} \delta \leq c_1^{-1} \) and \( \sqrt{k} \gamma_n \leq c_2^{-1} \) for any integer \( k \leq k_0(\delta, n) \), from (19) we have

\[
\| x_k^{\delta,n} - x^* \|^2 - \| P_n x_0 - x^* \|^2 \leq \frac{(1 + \eta)^2}{1 - 2\eta} \left( c_1^{-1} + C_0 c_2^{-1} \| (I - P_n)x^* \| \right)^2.
\]

Since \( (1 + \eta)^2/(1 - 2\eta)c_1^2 \leq (1/4)\rho^2 \), we can choose \( n \) large enough such that

\[
\| x_k^{\delta,n} - x^* \|^2 - \| P_n x_0 - x^* \|^2 \leq \frac{\delta}{16} \rho^2.
\]

This gives \( \| x_k^{\delta,n} - x^* \|^2 \leq \frac{\rho^2}{16} \) if \( n \) is large enough. Hence \( x_k^{\delta,n} \in B_{3\rho/4}(x^*) \).

To proceed further, a slight strengthened version of Assumption 2.1 (ii) is needed.

Assumption 3.1. There is a constant \( K_0 \) such that for every pair \( x, z \in B_{\rho}(x_0) \) there holds

\[
\| F(x) - F(z) - F'(z)(x - z) \| \leq K_0 \| x - z \| \| F'(z)(x - z) \|. \tag{20}
\]
We note that Assumption 3.1 implies Assumption 2.1(ii) if \( \rho > 0 \) is such that 
\[ K_0 \rho/(1 - K_0 \rho) \leq \eta. \]

**LEMMA 3.2.** Under Assumptions 2.1 and 3.1, if (1) is solvable in \( B_{\rho/4}(x_0) \), 
\[ K_0 \rho < (2\sqrt{2} - 4)/3, \] and \( c_i \) is chosen such that \( c_i \geq 2(1 + \eta)/\rho \sqrt{1 - 2\eta} \), 
then for all integers \( 0 < k \leq k_0(\delta, n) \), we have 
\[
\| F(x_k^{\delta, n}) - y_\delta \| \leq \| F(x_{k-1}^{\delta, n}) - y_\delta \|
\]
provided that \( n \) is large enough and \( \delta \) is suitably small.

**Proof.** Let \( w_k := F(x_k^{\delta, n}) - F(x_{k-1}^{\delta, n}) - F'(x_{k-1}^{\delta, n})(x_k^{\delta, n} - x_{k-1}^{\delta, n}) \). Then the definition of \( x_k^{\delta, n} \) gives 
\[
\| F(x_k^{\delta, n}) - y_\delta \|^2 - \| F(x_{k-1}^{\delta, n}) - y_\delta \|^2 
= 2(F(x_k^{\delta, n}) - F(x_{k-1}^{\delta, n}), F(x_k^{\delta, n}) - y_\delta) + \| F(x_k^{\delta, n}) - F(x_{k-1}^{\delta, n}) \|^2 
= -2\| x_k^{\delta, n} - x_{k-1}^{\delta, n} \|^2 + 2(w_k, F(x_{k-1}^{\delta, n}) - y_\delta) 
+ \| F(x_k^{\delta, n}) - F(x_{k-1}^{\delta, n}) \|^2. \tag{21}
\]

By utilizing Assumptions 3.1 and Assumption 2.1(i) we have 
\[
(w_k, F(x_{k-1}^{\delta, n}) - y_\delta) 
\leq K_0 \| x_k^{\delta, n} - x_{k-1}^{\delta, n} \| \| F'(x_{k-1}^{\delta, n})(x_k^{\delta, n} - x_{k-1}^{\delta, n}) \| \| F(x_{k-1}^{\delta, n}) - y_\delta \| 
\leq K_0 \| x_k^{\delta, n} - x_{k-1}^{\delta, n} \|^2 \delta + \| F(x_{k-1}^{\delta, n}) - F(x^1) \|, 
\]
where \( x^1 \) is a solution of (1) in \( B_{\rho/4}(x_0) \). Taking into account Assumption 2.1(ii) and noting that 
\( \| x_{k-1}^{\delta, n} - x^1 \| \leq \frac{1}{2} \rho \) which follows from Lemma 3.1, we have 
\[
\| F(x_{k-1}^{\delta, n}) - F(x^1) \| \leq \frac{1}{1 - \eta} \| F'(x^1)(x_{k-1}^{\delta, n} - x^1) \| \leq \frac{1}{1 - \eta} \| x_{k-1}^{\delta, n} - x^1 \| 
\leq \frac{1}{1 - \eta} \frac{3 \rho}{4} \leq \frac{3}{2} \rho.
\]

Therefore 
\[
(w_k, F(x_{k-1}^{\delta, n}) - y_\delta) \leq \frac{1}{2} K_0(2 \delta + 3 \rho) \| x_k^{\delta, n} - x_{k-1}^{\delta, n} \|^2. \tag{22}
\]
From (20) we have
\[ \left\| F(x_k^{\delta,n}) - F(x_{k-1}^{\delta,n}) \right\| \]
\[ \leq (1 + K_0\|x_k^{\delta,n} - x_{k-1}^{\delta,n}\|)\left\| F'(x_k^{\delta,n})(x_k^{\delta,n} - x_{k-1}^{\delta,n}) \right\| \]
\[ \leq (1 + \frac{3}{2}K_0\rho)\|x_k^{\delta,n} - x_{k-1}^{\delta,n}\|. \] 
(23)

Substituting (22) and (23) into (21), we therefore obtain
\[ \left\| F(x_k^{\delta,n}) - y_\delta \right\|^2 - \left\| F(x_{k-1}^{\delta,n}) - y_\delta \right\|^2 \]
\[ \leq - \left( 2 - K_0(2\delta + 3\rho) - (1 + \frac{3}{2}K_0\rho)^2 \right)\|x_k^{\delta,n} - x_{k-1}^{\delta,n}\|^2. \]

Since \( 2 - 3K_0\rho - (1 + \frac{3}{2}K_0\rho)^2 > 0 \), we have for sufficiently small \( \delta > 0 \) that
\[ \left\| F(x_k^{\delta,n}) - y_\delta \right\|^2 - \left\| F(x_{k-1}^{\delta,n}) - y_\delta \right\|^2 \leq 0 \]
and hence the assertion follows. \( \blacksquare \)

**Lemma 3.3.** Under the assumptions in Lemma 3.2, let \( x^l \) be a solution of (1) in \( B_{\rho/l}(x_0) \); then for all integers \( 0 \leq l < k_0(\delta, n) \) we have
\[ \|x_l^{\delta,n} - x^l\|^2 \leq \|x_l^{\delta,n} - x^l\|^2 + 3(1 + \eta)k\|F(x_l^{\delta,n}) - y_\delta\|^2 \]
\[ + (1 + \eta)k\|F(P_n x^l) - y_\delta\|^2 \] 
(24)
provided that \( n \) is large enough and \( \delta \) is suitably small.

**Proof.** From (13) and Assumption 2.1(ii) we have for \( 0 \leq j < k_0(\delta, n) \) that
\[ \|x_j^{\delta,n} - x^l\|^2 \]
\[ \leq \|x_j^{\delta,n} - x^l\|^2 + 2\left( F'(x_j^{\delta,n})(x_j^{\delta,n} - P_n x^l), F(x_j^{\delta,n}) - y_\delta \right) \]
\[ = \|x_j^{\delta,n} - x^l\|^2 + 2\left( F(x_j^{\delta,n}) - F(P_n x^l), F(x_j^{\delta,n}) - y_\delta \right) \]
\[ + 2\left( F(P_n x^l) - F(x_j^{\delta,n}) - F'(x_j^{\delta,n})(P_n x^l - x_j^{\delta,n}), F(x_j^{\delta,n}) - y_\delta \right) \]
\[ \leq \|x_{j+1}^{\delta,n} - x^l\|^2 + 2(1 + \eta)\left\| F(x_j^{\delta,n}) - F(P_n x^l) \right\| \left\| F(x_j^{\delta,n}) - y_\delta \right\| \]
\[ \leq \|x_{j+1}^{\delta,n} - x^l\|^2 + 2(1 + \eta)\left\| F(x_j^{\delta,n}) - y_\delta \right\|^2 \]
\[ + 2(1 + \eta)\left\| F(P_n x^l) - y_\delta \right\| \left\| F(x_j^{\delta,n}) - y_\delta \right\| \]
\[ \leq \|x_{j+1}^{\delta,n} - x^l\|^2 + 3(1 + \eta)\left\| F(x_j^{\delta,n}) - y_\delta \right\|^2 \]
\[ + (1 + \eta)\left\| F(P_n x^l) - y_\delta \right\|^2. \]

By using Lemma 3.2, this gives (24). \( \blacksquare \)
Now we can give the main result of this section.

**Theorem 3.1.** Under the assumptions in Lemma 3.2, if $\tau > \frac{2(1 + \eta)}{1 - \omega}$ then for small $\delta > 0$ and large $n$ there holds

$$
\|x_{\delta,n} - x^*\| \leq C_2 \left( \inf_{0 \leq k \leq k(\delta,n)} \left\{ \|x_{k,n} - x^*\| + \sqrt{k} \delta \right\} + \| (I - P_n) x^* \| \right)
$$

with $k(\delta,n)$ chosen by Rule 2.1, where $C_2 := \max\{ C_1, (\sqrt{3} \tau + 1)\sqrt{1 + \eta}, \sqrt{1 + \eta} C_0 c_2^{-1} \}$.

**Proof.** To prove this assertion, we need only to show that for all integers $0 \leq k \leq k_0(\delta,n)$ there holds

$$
\|x_{k,n} - x^*\| \leq C_2 \left(\|x_{k,n} - x^*\| + \sqrt{k} \delta + \| (I - P_n) x^* \| \right). \tag{25}
$$

Considering Lemma 2.1, we obtain (25) immediately if $k \leq k(\delta,n)$. If $k > k(\delta,n)$, then we can use Lemma 3.3 to obtain

$$
\|x_{k(\delta,n)} - x^*\| \leq C_2 \left(\|x_{k,n} - x^*\| + \sqrt{k} \delta + \| (I - P_n) x^* \| \right).
$$

From (11) and (17) and noting that $\sqrt{k} \gamma_n \leq c_2^{-1}$ it follows

$$
\|x_{k(\delta,n)} - x^*\| \leq \|x_{k,n} - x^*\| + \sqrt{3} \tau + 1)\sqrt{1 + \eta} \sqrt{k} \delta + \sqrt{1 + \eta} C_0 c_2^{-1} \| (1 - P_n) x^* \|,
$$

and we again obtain (25). \( \blacksquare \)

### 4. Rates of Convergence

In Section 2 we have considered the convergence of $x_{\delta,n} \in \Omega_k(\delta,n)$, but we cannot obtain any information on the rates of convergence. In fact, the rates of convergence can be arbitrarily slow. Therefore, to guarantee a suitable rate, some additional assumptions should be imposed on $x^*$ and these conditions are called “source conditions.” The following one is frequently used in nonlinear ill-posed problems:

there is a $\nu > 0$ and an element $\omega \in A\left(F'(x^*)\right)^\perp \subset X$ such that

$$
x_0 - x^* = \left(F'(x^*)^* F'(x^*)\right)^{\nu/2} \omega. \tag{26}
$$

To proceed, an additional assumption on $F$ is required.
Assumption 4.1. There is a constant $K_1$ such that for every $(x, z, y) \in B_p(x_0) \times B_p(x_0) \times Y$ there exists $l(x, z, y) \in Y$ satisfying $\|l(x, z, y)\| \leq K_1 \|x\| \|x - z\|$ such that

$$(F'(x)^* - F'(z)^*)y = F'(z)^*l(x, z, y).$$

One can consult [4, 5, 14] for some illustrative examples. We point out that Assumption 4.1 has the consequence (see [5])

$$\|F(x) - F(z) - F'(z)(x - z)\| \leq \frac{1}{2}K_1\|x - z\|\|F'(z)(x - z)\|$$

(27)

for all $x, z \in B_p(x_0)$ which is just (20) with $K_0 = K_1/2$. Furthermore, by denoting $\tilde{\gamma}_n := \|(I - P_n)F'(x^1)^*\|$, we have from Assumption 4.1 that (see [5])

$$\|(I - P_n)F'(x)^*\| \leq (1 + 2K_1\rho)\tilde{\gamma}_n$$

for all $x \in B_p(x_0)$. Since $\tilde{\gamma}_n \to 0$ when $F'(x^1)$ is compact (see [3]), we thus have shown that Assumption 2.1(iii) can be guaranteed if Assumption 4.1 is fulfilled and $F'(x^1)$ is compact.

Since the following derivation of the convergence rates under condition (26) involves some estimates on fractional powers of linear operators, we recall some well known results in the following.

**Lemma 4.1.** Let $\nu > 0$, and let $B : X \to Y$ be a bounded linear operator with the property $\|B\| \leq 1$. Then

$$\left\| \sum_{j=0}^{k-1} (I - B^*B)^j B^* \right\| \leq \sqrt{k}, \quad \|(I - B^*B)^k (B^*B)^k\| \leq (k + 1)^{-\nu}.$$  

(28)

If we use $P$ to denote an ortho-projector on $X$ and set $A := BP$, then for $\nu \leq 1$ there hold

$$\|(I - P)(B^*B)^{\nu/2}\| \leq B(I - P)^{\nu},$$

(29)

$$\|(A^*A)^{\nu/2} - (B^*B)^{\nu/2}\| \leq C_\nu\|(I - P)B^*\|^{\nu},$$

(30)

where $C_\nu$ is a constant depending on $\nu$ only.

**Proof.** Please refer to [9] for (28) and to [12] for (29) and (30). \]

Now we can state the main result on the rates of convergence for $x^k_{\delta,n}$. In the following $C$ is always used to denote a generic constant independent of $\delta$ and $n$. 


THEOREM 4.1. Let Assumptions 2.1, 3.1, and 4.1 be fulfilled, and let $x^1$ be a solution of (1) in $B_{r/4}(x_0)$ such that (26) holds with $0 < \nu \leq 1$. Suppose $K_0 \rho < (2\sqrt{3} - 4)/3$ and $c_1 \geq (1 + \eta)/\rho \sqrt{1 - 2\eta}$ and let $x_0$ be chosen such that $\|x - P_n x_0\|/\log \gamma_n \to 0$ as $n \to \infty$. If $\|\omega\|$ is suitable small, then for sufficiently small $\delta$ and suitable large $n$ there holds
\[
\|x_{\delta,n} - x^1\| \leq C\left(\delta^{\nu/(1+\nu)} + \gamma_n^\nu + \|(I - P_n)x_0\|\right) \quad (31)
\]
for the $k(\delta,n)$ determined by Rule 2.1 with $\tau > \frac{2(1+\nu)}{1-\frac{2\nu}{\nu+1}}$.

The upper bound provided by (31) suggests that Landweber iteration combined with Rule 2.1 defines a regularization method of optimal order for each $0 < \nu \leq 1$ at least with respect to $\delta$. We emphasize that the requirement $\|x - P_n x_0\|/\log \gamma_n \to 0$ is not a restrictive one. The severe restriction on $x_0$ is (26). However, (26) cannot be dropped since it is necessary for the expected convergence rates as indicated in [3] for regularization methods of linear ill-posed problems. The proof of Theorem 4.1 is based on the following estimates.

**Lemma 4.2.** Under the assumptions in Theorem 4.1, if $\|\omega\|$ is suitable small, then for sufficiently small $\delta$ and suitable large $n$ there hold
\[
\|e_k\| \leq C\left((k + 1)^{-\nu/2} + \gamma_n^\nu + \|(I - P_n)x_0\| + \sqrt{k}\delta\right), \quad (32)
\]
and
\[
\|\mathcal{A}e_k\| \leq C\left((k + 1)^{-1/2} + \gamma_n^{-1} + \gamma_n + \|(I - P_n)x_0\| + \delta\right) \quad (33)
\]
for all integer $0 \leq k \leq (c_1\delta^{1/(1+\nu)} + c_2\gamma_n)^{-2}$, where $e_k := x^1 - x_{\delta,n}$ and $\mathcal{A} := F(x^1)$.

**Proof of Theorem 4.1.** From (26), (29), and Assumption 2.1(iii) we have
\[
\|(I - P_n)x^1\| \leq C_0 \gamma_n^\nu \|\omega\| + \|(I - P_n)x_0\|. \quad (34)
\]
If we choose $\tilde{k}(\delta,n)$ to be the largest integer satisfying $\tilde{k}(\delta,n) \leq (c_1\delta^{1/(1+\nu)} + c_2\gamma_n)^{-2}$, then $\tilde{k}(\delta,n) \leq k_0(\delta,n)$ for suitable small $\delta$. Therefore we can use Theorem 3.1 and Lemma 4.2 to obtain
\[
\|x_{\delta,n} - x^1\|
\leq C\left((\tilde{k}(\delta,n) + 1)^{-\nu/2} + \gamma_n^\nu + \|(I - P_n)x_0\| + \sqrt{\tilde{k}(\delta,n)}\delta\right)
\leq C\left(\delta^{\nu/(1+\nu)} + \gamma_n^\nu + \|(I - P_n)x_0\|\right)
\]
and the proof is complete. \[\Box\]
Finally we conclude this section by presenting the proof of Lemma 4.2.

**Proof of Lemma 4.2.** The technique developed in [4] can be used to prove (32) and (33). Below we just give a sketch of the proof.

With the notation $\mathcal{A}_n := F'(x^* P_n)$, we first use the definition of $x_{k,n}^*$ and Assumption 4.1 to get

$$e_k = e_{k-1} + P_n F'(x_{k-1,n})^* (F(x_{k-1,n}) - y_0)$$

$$= (I - \mathcal{A}_n^* \mathcal{A}_n) e_{k-1} + \mathcal{A}_n^* \mathcal{A}_n (P_n - I) x^1$$

$$+ \mathcal{A}_n^* \left( F(x_{k-1,n}) - F(x^1) - \mathcal{A}_n^* (x_{k-1,n} - x^1) \right) + \mathcal{A}_n^* (y - y_0)$$

$$+ \mathcal{A}_n^* (x_{k-1,n} - x^1, F(x_{k-1,n}) - y_0).$$

This together with Assumption 2.1(ii), Assumption 4.1, and (27) gives for all integers $0 \leq k \leq (c_1 \delta^{1/(1+\nu)} + c_2 \gamma_n)^{-1}$ the expressions

$$e_k = (I - \mathcal{A}_n^* \mathcal{A}_n)^k e_0 + \sum_{j=0}^{k-1} (I - \mathcal{A}_n^* \mathcal{A}_n)^j \mathcal{A}_n^* z_{k-j-1}$$

$$+ \left( \sum_{j=0}^{k-1} (I - \mathcal{A}_n^* \mathcal{A}_n)^j \mathcal{A}_n^* \right) \left( \mathcal{A}_n^* (P_n - I) x^1 + (y - y_0) \right)$$

(35)

and

$$\mathcal{A} e_k = \mathcal{A}_n (I - \mathcal{A}_n^* \mathcal{A}_n)^k e_0 + \mathcal{A}_n (I - P_n) (I - \mathcal{A}_n^* \mathcal{A}_n)^k e_0$$

$$+ \sum_{j=0}^{k-1} (I - \mathcal{A}_n^* \mathcal{A}_n)^j \mathcal{A}_n^* z_{k-j-1}$$

$$+ (I - (I - \mathcal{A}_n^* \mathcal{A}_n)^k) (\mathcal{A}_n^* (P_n - I) x^1 + (y - y_0)),$$

(36)

where

$$\|z_{k-j-1}\| \leq \frac{\delta}{2} K_1 \|e_{k-j-1}\| \|\mathcal{A} e_{k-j-1}\| + K_1 \delta \|e_{k-j-1}\|.$$  

(37)

By (26), (29), and (30) one can show that

$$\left\| (I - \mathcal{A}_n^* \mathcal{A}_n)^k e_0 \right\| \leq \left( C \gamma_0 + (k + 1)^{1/2} \right) \|\omega\| + \left\| (I - P_n) x_0 \right\|,$$

(38)

$$\left\| \mathcal{A}_n (I - \mathcal{A}_n^* \mathcal{A}_n)^k e_0 \right\| \leq \left( C \gamma_0 (k + 1)^{-1/2} \gamma_0 + (k + 1)^{-1/2} \right) \|\omega\|$$

$$+ (k + 1)^{-1/2} \left\| (I - P_n) x_0 \right\|,$$

(39)
Using Lemma 4.1 to estimate (35) and (36), then substituting (38) and (39) into the results, noting (34) and $\sqrt{k} \gamma_n \leq c_2^{-1}$, and applying Young’s inequality to some terms, we can show that there is a constant $C_n'$ depending only on $\nu$ such that

$$
\|e_k\| \leq C_n' \left( (\gamma_n^+ + (k + 1)^{-\nu/2}) \|\omega\| + \| (I - P_n) x_0 \| \right) + \sqrt{k} \delta \\
+ \sum_{j=0}^{k-1} (j + 1)^{-1/2} \|z_{k-j} - 1\|,
$$

(40)

$$
\|\mathcal{A}e_k\| \leq C_n' \left( (\gamma_n^+ + (k + 1)^{-\nu/2}) \|\omega\| \\
+ (\gamma_n + (k + 1)^{-1/2}) \| (I - P_n) x_0 \| \right) \\
+ \delta + \sum_{j=0}^{k-1} (j + 1)^{-1} \|z_{k-j} - 1\|.
$$

(41)

Based on (37), (40), and (41), and the assumption that $\| (I - P_n) x_0 \| / \log \gamma_n \to 0$ as $n \to \infty$, with some tedious but elementary calculations we can show by induction that if $\|\omega\|$ is suitable small then (32) and (33) are valid for suitable small $\delta$ and sufficiently large $n$. Therefore the proof is complete. 

5. NUMERICAL EXPERIMENT

In this section we present a numerical example to evaluate the performance of the Landweber iteration endowed with Rule 2.1.

In order to implement (6) numerically, we need to calculate the correction term $r = P_n F'(x)^\psi (F(x) - y_\delta)$ (with $x = x_\delta$), which can be carried out from the variational formulation

$$
(r, \psi) = (F(x) - y_\delta, F'(x) \psi), \quad \psi \in X_n.
$$

Suppose $\{\psi_1, \ldots, \psi_N\}$ is a basis of $X_n$ (we assume dim $X_n = N$ here) and expand $r$ as

$$
r = \sum_{i=1}^{N} r_i \psi_i,
$$

then the vector $r := (r_1, \ldots, r_N)^T$ can be obtained by solving the linear algebraic system

$$
Lr = d,
$$
where
\[ L := \left( \left( \psi_i, \psi_j \right) \right)_{i,j=1}^N, \quad d^T := \left( \left( F(x) - y_\delta, F'(x) \psi_j \right) \right)_{j=1}^N. \]

The vector \( d \) involves \( F(x) \) and \( F'(x) \psi \), which in many cases satisfy some differential equation problems which can be solved by finite element methods.

In the following we consider the identification of the coefficient \( a \) in the two-point boundary value problem
\[
\begin{aligned}
- u'' + au &= f, & t \in (0,1) \\
u(0) &= u_0, & u(1) = u_1
\end{aligned}
\tag{42}
\]
from the measurement data \( u_\delta \) of the state variable \( u \), where \( u_0, u_1 \), and \( f \in L^2[0,1] \) are given. Now the nonlinear operator \( F : D(F) \subset L^2[0,1] \mapsto L^2[0,1] \) is defined as the parameter-to-solution mapping \( F(a) = u(a) \) with \( u(a) \) being the unique solution of (42). \( F \) is well-defined on
\[ D(F) := \{ a \in L^2[0,1] : \| a - \hat{a} \|_{L^2} \leq \gamma \text{ for some } \hat{a} \geq 0 \text{ a.e.} \} \]
with some \( \gamma > 0 \). Moreover, \( F \) is Fréchet differentiable, and the Fréchet derivative and its adjoint are given by
\[
F'(a) h = -A(a)^{-1} \left( hu(a) \right), \\
F'(a)^* w = -u(a) A(a)^{-1} w,
\]
where \( A(a) : H^2 \cap H^1_0 \mapsto L^2 \) is defined by \( A(a)u = -u'' + au \). Assumptions 3.1 and 4.1 have been shown in [4, 14] if \( |u(a')(t)| \geq \kappa > 0 \) for all \( t \in [0,1] \).

In the following computation we take \( X_n \) to be the subspace of piecewise linear splines on a uniform grid with subinterval length \( n^{-1} \) and let \( P_n \) be the ortho-projector of \( L^2[0,1] \) onto \( X_n \). From [10] we see that Assumption 2.1(iii) is valid with \( \gamma_n = n^{-2} \). Furthermore, if \( x_0 \in H^s[0,1] \) the Sobolev space of order \( s \) with \( s = 1 \) or 2, then there exists a constant \( C_s \) such that (see [1])
\[ \| (I - P_n) x_0 \|_{L^2} \leq C_s n^{-s} \| x_0 \|_{H^s}. \]

We present the numerical results in Table I. We remark that the differential equation problems we meet during computation are always solved approximately by a finite element method on the subspace of piecewise linear splines on a uniform grid with subinterval length \( \frac{1}{2n} \). All computations were performed by the Matlab software package.
TABLE I
Numerical Results for Example 5.1

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$n$</th>
<th>$k(\delta, n)$</th>
<th>Error $= | a_{k(\delta, n)} - a^1 |_2$</th>
<th>Error $/(\delta^{1/2} + n^{-2})$</th>
</tr>
</thead>
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<tr>
<td>0.01</td>
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<td>96</td>
<td>0.2205</td>
<td>2.1221</td>
</tr>
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<td>32</td>
<td>101</td>
<td>0.2123</td>
<td>2.1020</td>
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<td>2.0973</td>
</tr>
<tr>
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<td>0.6682</td>
</tr>
<tr>
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<td>0.6623</td>
</tr>
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</tr>
<tr>
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<td>0.0032</td>
<td>0.2933</td>
</tr>
<tr>
<td></td>
<td>48</td>
<td>679</td>
<td>0.0031</td>
<td>0.2992</td>
</tr>
</tbody>
</table>

Example 5.1. This example is performed to test the theoretical assertions given in Theorem 4.1. Here we estimate $a$ in (42) by assuming $f = 1 + t^2$ and $u_0 = u_1 = 1$. If $u(a^1) = 1$ then the true solution is $a^1 = 1 + t^2$. In our computation, we used the first guess $a_0 = 1 + t^2 + 2(t - 2t^3 + t^5)$, and instead of $u(a^1)$ we used the special perturbation $u \approx 1 + \sqrt{2} \sin(2\pi t/\delta)$ with high frequency. Clearly $\| u_0 - u(a^1) \|_2 \leq \delta$. It is easy to know that $a_0 - a^1 \in R(F(a^1)^*) = R((F'(a^1)^*F(a^1))^{1/2})$. Since $\| (I - P_n)a_0 \| \leq O(n^{-2})$, the rates of convergence we can expect should be $O(\delta^{1/2} + n^{-2})$.

Table I contains the results for the discrete Landweber iteration (6) which is terminated by Rule 2.1 with $c_1 = c_2 = 1$ and $\tau = 2.1$. In order to indicate the dependence of the convergence rates on noise level and discretized level, different values of $\delta$ and $n$ are selected. The rates in Table I coincide with Theorem 4.1 very well.

REFERENCES


