On the Towers of Hanoi problem with multiple spare pegs

Jeyakesavan Veerasamy and *Ivor Page*
Computer Science Program, EC 31
The University of Texas at Dallas
Richardson, TX 75083-0688, USA

{veerasam|ivor}@utdallas.edu

Abstract

We analyze a solution to a variant of the Towers of Hanoi problem, in which multiple spare pegs are used to move the disks from the source to the destination peg. We also discuss the interesting relation between the number of disks and the total number of disk moves when the number of spare pegs is a function of number of disks.

Keywords
Towers of Hanoi, analysis of algorithms, recurrence relations, recursion

C.R. Category: F.2.2.

1 Introduction

The Towers of Hanoi problem is well-known for its nice recursive solution. In the recent years, there has been increasing interest in several variants of the original problem. References [1-7] present a few such interesting variations. In [3], X.M.Lu discussed an optimal algorithm for the Towers of Hanoi Problem with multiple spare pegs. But it is difficult to compare his final expression for the number of disk moves with the solution of the original problem and appreciate the improvement in the number of disk moves. In this paper, we derive a simple upper bound which enables us to understand the usefulness of additional spare pegs. We also discuss the number of disk moves required, when the number of spare pegs is a function of number of disks.

The main results of [3] in slightly simplified form follows. Let k be the number of spare pegs ($k+2$ is the total number of pegs) and n be the number of disks to be moved from the source peg to the destination peg. The total number of disk moves $Q(k,n)$ is given by the following recurrence relation:

$$\begin{aligned} Q(k,0) &= 0 \\ Q(k,n) &= Q(k,n-1)+2^l \end{aligned}$$

where l is defined by the inequality, $\binom{k+l-1}{l-1} < n \le \binom{k+l}{l}$, and n is said to be at the level l. Fig.1 shows the piecewise linear nature of $Q(k,n)$ verses n. It is clear that the total number of disk moves increases by 2^i for each disk added at level i. Using k spare pegs and the levels

$\leq l$, at the maximum $N(k,l) = \binom{k+l}{l}$ disks can be transferred. The following recurrence relation explains the algorithm.

$$\begin{aligned} N(0,l) &= 1 \\ N(k,0) &= 1 \\ N(k,l) &= N(k,l-1) + N(k-1,l) \end{aligned}$$

To transfer $N(k,l)$ disks from the source peg to the destination peg, first transfer $N(k,l-1)$ disks to one of the spare pegs using k spare pegs (the destination peg is a spare peg in this phase), transfer the remaining disks $N(k-1,l)$ to the destination peg using $k-1$ spare pegs, and finally transfer $N(k,l-1)$ disks from the spare peg to the destination peg using k spare pegs (the source peg is a spare peg in this phase). Similarly, the following equation is the result of applying this 3 stage process to $N(k,l+1)$ disks.

$$N(k,l+1) = N(k,l) + N(k-1,l+1)$$

For all n such that $N(k,l) < n \leq N(k+1,l)$, choose n_1, n_2 such that $n = n_1 + n_2$, $N(k,l-1) \leq n_1 < N(k,l)$ and $N(k-1,l) \leq n_2 < N(k-1,l+1)$ and follow the same method to transfer n disks. The number of disks added in level i is $N(k,i) - N(k,i-1) = N(k-1,i)$. So the total number of disk moves is given by

$$Q(k,n) = \sum_{i=0}^{l} N(k-1,i)2^i + (n - N(k,l))2^{l+1}$$

2 Upperbound

Now we will derive a simple upper bound for the total number of moves. In the last section, we found that for each disk added at level i, the total number of moves increases by 2^i. So if n disks are moved using k spare pegs and the levels $\leq l$, a trivial upper bound for the number of disk moves is $n2^l$. We have to express l as a function of k and n. Let us consider all n values at the level l.

$$\begin{aligned} N(k,l-1) < \quad & n && \leq N(k,l) \\ \binom{k+l-1}{l-1} < \quad & n && \leq \binom{k+l}{l} \\ \frac{l(l+1)\cdots(l+k-1)}{k!} < \quad & n && \leq \frac{(l+1)(l+2)\cdots(l+k)}{k!} \\ \sqrt[k]{l(l+1)\cdots(l+k-1)} < \quad & \sqrt[k]{k!n} && \leq \sqrt[k]{(l+1)(l+2)\cdots(l+k)} \\ l < \quad & \sqrt[k]{k!}\sqrt[k]{n} && \leq l+k \end{aligned}$$

$$Q(k,n) < n2^l < n2^{\sqrt[k]{k!}\sqrt[k]{n}} = n2^{c\sqrt[k]{n}}$$

Even though our equation does not give the exact number of moves, comparing this upper bound with the solution of the original problem $Q(1,n) = 2^n - 1$, we can realize the effect of the additional spare pegs.

Till now, we assumed a constant number of spare pegs and found that the total number of moves is exponential in n. In the following sections, we analyze the problem in which the number of spare pegs is a function of n.

3 When $k = \lceil cn^\alpha \rceil$

Theorem 1 *When the number of spare pegs $k = \lceil cn^\alpha \rceil$ where c and α are constants, the total number of disk moves is linear in n.*

Proof: When $\alpha > 1$, it is easy to see that for $n > n_0$, $cn^\alpha \geq n-1$. Once the number of spare pegs $k \geq n-1$, $Q(n,k) = 2n-1$. Hence the number of moves is linear in n.

Now we consider $\alpha \leq 1$. We have already found a simple upper bound $n2^l$ where n is the number of disks and l is the level of n. We will show this bound is linear in n, *i.e.* we prove that l is bounded by a constant t which is a function of c and α and does not depend on n.

$$k \geq cn^\alpha \Rightarrow n \leq (k/c)^{1/\alpha} = (k/c)^\beta = (1/c)^\beta k^\beta = dk^\beta$$

where $\beta = 1/\alpha$ and $d = 1/c^\beta$.

Lemma 1 *If the following inequalities hold for any constants k_0 and t,*

$$dk_0^\beta \leq N(k_0, t) \tag{1}$$

$$\frac{(k_0+1)^\beta}{k_0^\beta} \leq \frac{(k_0+1+t)}{(k_0+1)} \tag{2}$$

then $dk^\beta \leq N(k,t)$ for any $k \geq k_0$.

Proof: We will prove this lemma by induction. By inequality 1, the lemma is trivially true for $k = k_0$. Multiplying the sides of the inequalities, we get

$$dk_0^\beta \frac{(k_0+1)^\beta}{k_0^\beta} \leq \frac{(k_0+t)!}{k!t!} \frac{(k_0+1+t)}{k_0+1}$$

$$d(k_0+1)^\beta \leq \frac{(k_0+1+t)!}{(k_0+1)!t!} = N(k_0+1, t)$$

So the lemma is true for $k = k_0 + 1$.

Let us assume that the lemma holds good for any $k < k_0 + m$ where $m > 1$. So

$$d(k_0+m-1)^\beta \leq N(k_0+m-1, t) \tag{3}$$

Adding $m-1$ to denominator and numerator of inequality 2, we get

$$\begin{aligned} \frac{(k_0+1)^\beta + m - 1}{k_0^\beta + m - 1} &\leq \frac{(k_0+m+t)}{(k_0+m)} \\ \frac{(k_0+m)^\beta}{(k_0+m-1)^\beta} &\leq \frac{(k_0+m+t)}{(k_0+m)} \end{aligned}$$

Combining this inequality with inequality 3, we get $d(k_0+m)^\beta \leq N(k_0+m, t)$. So the lemma is true for $k = k_0 + m$. ■

If we set $k_0 = 1$ in lemma 1, we get

$$\begin{aligned} d &\leq t+1 \\ 2^\beta &\leq \frac{(t+2)}{2} \end{aligned}$$

So for $t = \lceil max(d-1, 2^{\beta+1} - 2) \rceil$, the lemma holds true. So the total number of disk moves is $\leq n2^l \leq n2^t$ where l is the actual level of n, and $t = \lceil max(d-1, 2^{\beta+1}-2) \rceil$. Fig.2 shows the actual relationship between k and l for different α and c values. It is interesting to note that as k increases, l rapidly increases, reaches a maximum l_{max} and drops down, and then remains almost a constant. Changing the value of c has significant effect on l, only for small k values. For a fixed α, l shoots up more for $c < 1$, and l shoots up less for $c > 1$, then l settles to an almost constant value, irrespective of c . ■

4 When k is logarithmic in n

Theorem 2 *When the number of spare pegs $k = \lceil c \log n \rceil$, where c is a constant, the total number of disk moves is $\leq 2n^d$ where $d = 1 + c(\sqrt[c]{2} - 1)$.*

Proof:

$$\begin{aligned} k &\leq c \log n \\ n &\geq 2^{k/c} \leq N(k,l) = \binom{k+l}{k} \\ \sqrt[c]{2}^k &\leq \frac{(l+1)(l+2)\cdots(l+k)}{k!} \end{aligned}$$

Now we have to decide the relationship between k and l. This inequality will not hold good for any constant l because, as k increases, the rhs is not able to keep up with growth of the lhs. We rewrite the equation introducing one more variable x,

$$(1+\frac{x}{k})^k = (\sqrt[c]{2})^k \leq (1+l)(1+\frac{l}{2})(1+\frac{l}{3})\cdots(1+\frac{l}{k})$$