Connected Morphological Operators and Filters for Binary Images

Henk J.A.M. Heijmans
Centre for Mathematics and Computer Science (CWI)
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Abstract

Connected morphological operators act on the level of the flat zones of an image, i.e., the connected regions where the grey-level is constant. For binary images, the flat zones are the foreground and background grains (connected components) of the image. The flat zones constitute a partition of the underlying space. A connected operator is an operator that makes this partition coarser. A grain operator is a connected operator that is uniquely specified by two grain criteria, one for the foreground and one for the background components. This paper contains a brief discussion of connected operators and grain operators with an emphasis on connected filters, all in the context of binary images.

1 Introduction

Connectivity, in all its manifestations, has always been an important notion in the field of image processing. This is even more true for methods from mathematical morphology because of their intrinsic topological and geometrical nature. A simple, but extremely important instance of a morphological operation based on connectivity is the *reconstruction* of a point marker inside a set by successive dilations [1]. About five years ago, the first systematic studies on connected operators by Serra, Salembier, Crespo, Schafer, and others [2, 3, 4, 5] appear in the literature. A major impulse to the current research on connected operators was given by the work of Vincent, providing for the first time efficient algorithms for grey-scale reconstruction [6] and the area opening [7]; see below.

A connected operator is an operator that acts on the level of the flat zones of an image, rather than on the level of individual pixels. By flat zone we mean a maximal connected region where the grey-level is constant. In the binary case this means that such an operator cannot break connected components (grains) of the foreground or the background. Connected operators cannot introduce new discontinuities and as such they are eminently suited for applications where contour information is important. An important subclass of the connected operators is formed by the so-called grain operators. Such operators are completely determined by two grain criteria, one for the foreground and one for the background.

In Section 2 we explain how to define connected sets by using adjacency relations. Thus we can speak of foreground and background components of a set, and, for a given set, such components (also called *grains*) constitute a partition of the underlying space. Then a connected operator is defined as an operator that coarsens the partition. This is all explained in Section 3. In Section 4 we present a formal definition of grain operators, and in Section 5 we show how to build grain filters and, more generally, connected filters.

2 Adjacency and connectivity

In this paper, we model a binary image as a subset of some overall space E, e.g. $\mathbb{R}^d$ or $\mathbb{Z}^d$.

Definition 1 *A binary relation $\sim$ on $E \times E$ is called an* adjacency relation *if it is reflexive ($x \sim x$ for every x) and symmetric ($x \sim y$ iff $y \sim x$).*

Given an adjacency relation on $E \times E$, we call $x_0, x_1, \ldots, x_n$ a *path* between the points x and y if $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$. Define $\mathcal{C} \subseteq \mathcal{P}(E)$ as the collection of all $C \subseteq E$ such that any two points in C can be connected by a path that lies entirely in C. In the literature [8, 9], $\mathcal{C}$ is called a *connectivity class*, and its elements are called *connected sets*. Throughout this paper we assume that there exists a path between any two points in E, that is, $E \in \mathcal{C}$.

On $E = \mathbb{Z}^2$, two well-known adjacency relations are 4-adjacency and 8-adjacency. If E is a metric space with metric d and $r \geq 0$, then the relation '$x \sim y$ if $d(x,y) \leq r$' defines an adjacency relation.

Given a set $X \subseteq E$, and a subset $C \subseteq X$, we say that C is a *connected component* or *grain* of X, denoted as $C \Subset X$, if C is connected and if there exists no connected subset of X that is strictly larger than C.

The operator γ_h defined by

$$\gamma_h(X) = \begin{cases} \text{grain of } X \text{ that contains } h, & \text{if } h \in X \\ \varnothing, & \text{if } h \notin X \end{cases}$$

is called *connectivity opening* (indeed, it is not difficult to show that γ_h is an opening [8, 9]). Now one can

0-8186-8183-7/97 $10.00 © 1997 IEEE

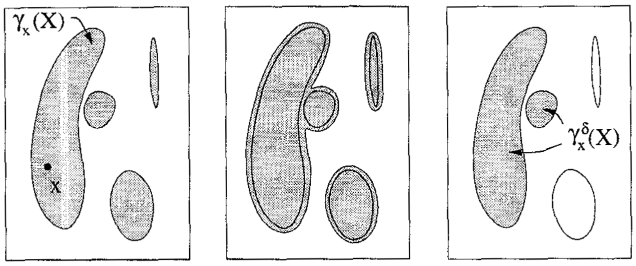

Figure 1: From left to right: a set X and the grain $\gamma_x(X)$; the dilation $\delta(X)$; the grain $\gamma_x^\delta(X)$.

define a *reconstruction operator* ρ as follows:

$$\rho(Y \mid X) = \bigcup_{h \in Y} \gamma_h(X). \tag{1}$$

If $X \cap Y = \varnothing$ then $\rho(Y \mid X) = \varnothing$. Observe that $\gamma_h(X) = \rho(\{h\} \mid X)$. The reconstruction $\rho(Y \mid X)$ (and hence the opening $\gamma_h(X)$) can be computed easily by means of a propagation algorithm.

An interesting method to build a new connectivity from an existing one is by means of dilation [9, 10].

Proposition 1 *Assume that $\sim$ is an adjacency relation on $E \times E$ which generates a connectivity class $\mathcal{C}$ and connectivity openings γ_x. Furthermore, let δ be an extensive dilation on $\mathcal{P}(E)$ such that $\delta(\{x\}) \in \mathcal{C}$, for every $x \in E$. Then $\overset{\delta}{\sim}$ defined by "$x_1 \overset{\delta}{\sim} x_2$ if there exist $y_1 \in \delta(\{x_1\})$ and $y_2 \in \delta(\{x_2\})$ such that $y_1 \sim y_2$", defines an adjacency relation as well, with connectivity class*

$$\mathcal{C}^\delta = \{X \subseteq E \mid \delta(X) \in \mathcal{C}\}$$

and connectivity openings given by

$$\gamma_x^\delta = \mathrm{id} \wedge \gamma_x \delta, \quad x \in E.$$

Furthermore, the equality

$$\delta \gamma_x^\delta = \gamma_x \delta$$

holds.

This proposition is illustrated in Fig. 1.

3 Connected operators

By a *partition* of the space E we mean a subdivision of this space into disjoint zones. A partition can be represented by a function $P : E \to \mathcal{P}(E)$ that satisfies $x \in P(x)$, for every $x \in E$, and $P(x) = P(y)$ or $P(x) \cap P(y) = \varnothing$, for any two points $x, y \in E$. Here $P(x)$ is the zone of the partition that contains the point x. The partition is called connected if every zone is connected.

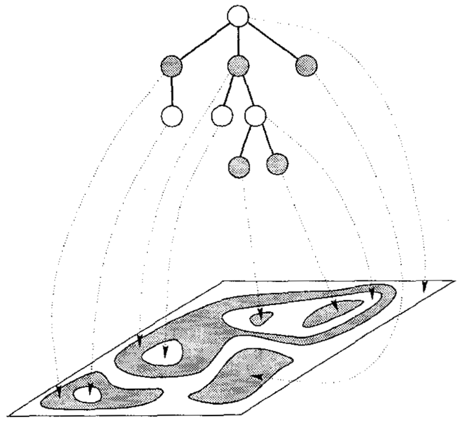

Figure 2: Zonal graph associated with a binary image.

The following relation defines a partial ordering on the collection of partitions of E:

$$P \sqsubseteq P' \text{ if } P'(h) \subseteq P(h), \text{ for every } h \in E.$$

We say that P is *coarser* than P'.

With every set $X \subseteq E$ one can associate a connected partition $P(X)$ defined as follows: $P(X)(h)$ is the connected component of X (if $h \in X$) or X^c (if $h \notin X$) that contains h.

Definition 2 *An operator ψ on $\mathcal{P}(E)$ is* connected *if the partition $P(\psi(X))$ is coarser than $P(X)$, for every $X \subseteq E$.*

By introducing the concept of a *zonal graph* (or *region adjacency graph*) one obtains additional insight into the behaviour of connected operators.

Every binary image $X \in \mathcal{P}(E)$ yields a coloured partition (with colours 0 and 1) of the underlying space E. Now, consider the parts of $P(X)$ as the vertices of a graph, two parts $C_1, C_2 \in P(X)$ being *adjacent* (denoted by $C_1 \sim C_2$) if $C_1 \cup C_2$ is a connected set. To determine X from this graph, we have to specify for every vertex whether it is a subset of X or X^c. To this end we use the colouring $I_X : P(X) \to \{0, 1\}$:

$$I_X(C) = \begin{cases} 1, & \text{if } C \Subset X \\ 0, & \text{if } C \Subset X^c \end{cases}$$

Note that, due to the fact that two adjacent vertices must have different colours, it suffices to specify the colour of only one vertex (this is not true in the greyscale case).

The triple $(P(X), \sim, I_X)$ is called the *zonal graph* associated with X; see Fig. 2 for an illustration.

Proposition 2 *If $E = \mathbb{Z}^2$ with 8-connectivity as adjacency relation, the the graph $(P(X), \sim)$ is a tree, for every set $X \subseteq \mathbb{Z}^2$.*

4 Grain operators

A connected operator that is of great interest is the *area opening* α_S given by:

$$\alpha_S(X) = \bigcup\{C \mid C \Subset X \text{ and } \mathrm{area}(C) \geq S\}. \quad (2)$$

It deletes all grains $C \Subset X$ with area less than S. The area opening is a typical representative of a class of connected operators that is known as *grain operators*. The area opening is determined by a criterion $u : \mathcal{C} \to \{0,1\}$ on the foreground grains, namely $u(C) = [\mathrm{area}(C) \geq S]$. Here we use the following convention: if S is a statement then $[S]$ denotes the Boolean value (0 or 1) indicating whether S is true or false. We call u a *grain criterion.*

Definition 3 *Given foreground and background criteria u and v, the operator*

$$\psi_{u,v}(X) = \bigcup\{C \mid (C \Subset X \textit{ and } u(C) = 1) \\ \textit{or } (C \Subset X^c \textit{ and } v(C) = 0)\} \quad (3)$$

is called a grain operator. *If v is identically 1, we write $\psi_{u,1}$.*

The criteria u and v are uniquely determined by the operator $\psi = \psi_{u,v}$:

$$u(C) = [C \subseteq \psi(C)] \text{ and } v(C) = [C \subseteq \psi^*(C)]$$

We say that u is increasing if $C \subseteq C'$ implies $u(C) \leq u(C')$. A useful (increasing) criterion is $u(C) = [C \ominus B \neq \varnothing]$, which gives the outcome 1 if some translate of B fits inside C. If B is connected, then $\psi_{u,1} = \breve{\alpha}$, where

$$\breve{\alpha}(X) = \rho(X \circ B \mid X),$$

the so-called *opening by reconstruction* [11, 12]. Various other criteria were given by Breen and Jones [13]. One can easily show that Boolean combinations of grain operators are grain operators. Also the negative of a grain operator is a grain operator [10]. Moreover, $(\psi_{u,v})^* = \psi_{v,u}$. In terms of the zonal graph, a grain operator ψ is characterised by the following property: the value $\psi(X)(h)$ is completely determined by the information provided by $X(h)$ and $P(X,h)$, 'stored' at the vertex $P(X,h)$ of the associated zonal graph. In other words, the recolouring of the zonal graph corresponding with a grain operator is entirely based upon local information stored at individual vertices; all other information, e.g. about adjacent vertices, is irrelevant.

5 Connected filters

In this section we summarise some results about connected filters; refer to [10] for a comprehensive discussion and proofs. Recall that an operator ψ on $\mathcal{P}(E)$ is called a (morphological) *filter* if it is increasing and idempotent. It is called a *strong filter* if $\psi(\mathrm{id} \wedge \psi) = \psi(\mathrm{id} \vee \psi) = \psi$.

Proposition 3 *If u is an increasing grain criterion, then $\alpha_u = \psi_{u,1}$ is an opening, and $\beta_u = \psi_{1,u}$ a closing. Furthermore, the duality relation*

$$\alpha_u^* = \beta_u$$

and the composition rules

$$\alpha_{u_1}\alpha_{u_2} = \alpha_{u_1 \wedge u_2} \quad \textit{and} \quad \beta_{u_1}\beta_{u_2} = \beta_{u_1 \wedge u_2}$$

hold.

We introduce the following notation: for $C_1, C_0 \in \mathcal{C}$ we write $C_1 \approx C_0$ if there exists $X \subseteq E$ such that $C_1 \Subset X$, $C_0 \Subset X^c$ and $C_1 \sim C_0$.

Proposition 4 *Let u, v be increasing grain criteria such that*

$$C_1 \approx C_0 \Rightarrow u(C_1) \vee v(C_0) = 1, \quad (4)$$

then

$$\psi_{u,v} = \alpha_u\beta_v = \beta_v\alpha_u;$$

in particular, $\psi_{u,v}$ is a strong grain filter.

Proposition 5 (*a*) *A supremum/infimum of strong grain filters is a strong grain filter.*
(*b*) *If $\psi_1, \psi_2, \ldots, \psi_n$ are strong grain filters, then $\psi = \psi_n\psi_{n-1}\cdots\psi_1$ is a strong connected filter.*

We consider alternating sequential filters. Recall the following notation: if ϕ_n, ψ_n are sequences of operators, then $(\psi\phi)_n$ denotes the composition

$$(\psi\phi)_n = \psi_n\phi_n\psi_{n-1}\phi_{n-1}\cdots\psi_1\phi_1$$

We say that a sequence of filters ψ_n satisfies the *absorption law* if

$$(\psi)_n(\psi)_m = (\psi)_n, \; n \geq m.$$

If in addition

$$(\psi)_m(\psi)_n = (\psi)_n, \; n \geq m,$$

then we say that the sequence ψ_n satisfies the *strong absorption law.*

Proposition 6 (*a*) *Let u_k, v_k, $k = 1, 2, \ldots, n$, be increasing grain criteria and $\alpha_k = \alpha_{u_k}, \beta_k = \beta_{v_k}$, then $(\beta\alpha)_n$ and $(\alpha\beta)_n$ are strong filters.*
(*b*) *Assume in addition that $u_1 \geq u_2 \geq \cdots \geq u_N$ and $v_1 \geq v_2 \geq \cdots \geq v_N$, then $(\beta\alpha)_n$ and $(\alpha\beta)_n$ satisfy*

$$(\beta\alpha)_n \leq (\alpha\beta)_n.$$

and they obey the strong absorption law.

Figure 3: Left to right and top to bottom: original image X (appr. 20% noise), and the area open-close filtered images $(\beta\alpha)_n(X)$ for $n = 1, 2, 3$.

Example 1 *Let E be the space $\mathbb{Z}^2$ endowed with 8-connectivity. Consider the area criterion $a_S(C) =$ [area$(C) \geq S$]. In Fig. 3 we illustrate the filters $(\beta\alpha)_n$ for $n = 1, 2, 3$, where $u_n = a_{S_n}$ and $S_1 = 5$, $S_2 = 20$, $S_3 = 100$. The noise-cleaning effect of these filters inside homogeneous regions is quite good; however, noise pixels adjacent to edges are not affected by these filters (as we have seen, this is a general property of connected operators). We make the following observation with regard to the filters $\omega_{S,T} = \beta_{a_T}\alpha_{a_S}$. It is not difficult to verify that condition (4) holds for the pair $u = a_S$, $v = a_T$ if $S, T \leq 8$. Thus, from Proposition 4 we get that*

$$\omega_{S,T} = \psi_{a_S,a_T} = \beta_{a_T}\alpha_{a_S} = \alpha_{a_S}\beta_{a_T}$$

is a strong grain filter if $S, T \leq 8$. If $S = T$, then $\omega_{S,T}$ is self-dual.

References

[1] J. Serra, *Image Analysis and Mathematical Morphology.* London: Academic Press, 1982.

[2] J. Crespo, *Morphological connected filters and intra-region smoothing for image segmentation.* PhD thesis, Georgia Institute of Technology, Atlanta, 1993.

[3] J. Crespo, J. Serra, and R. W. Schafer, "Theoretical aspects of morphological filters by reconstruction," *Signal Processing*, vol. 47, no. 2, pp. 201–225, 1995.

[4] P. Salembier and J. Serra, "Flat zones filtering, connected operators, and filters by reconstruction," *IEEE Transactions on Image Processing,* vol. 4, no. 8, pp. 1153–1160, 1995.

[5] J. Serra and P. Salembier, "Connected operators and pyramids," in *Image Algebra and Morphological Image Processing IV,* vol. 2030 of *SPIE Proceedings,* (San Diego), pp. 65–76, 1993.

[6] L. Vincent, "Morphological grayscale reconstruction in image analysis: efficient algorithms and applications," *IEEE Transactions on Image Processing,* vol. 2, pp. 176–201, 1993.

[7] L. Vincent, "Morphological area openings and closings for grey-scale images," in *Proceedings of the Workshop "Shape in Picture", 7-11 September 1992, Driebergen, The Netherlands* (Y.-L. O, A. Toet, D. Foster, H. J. A. M. Heijmans, and P. Meer, eds.), (Berlin), pp. 197–208, Springer, 1994.

[8] H. J. A. M. Heijmans, *Morphological Image Operators.* Boston: Academic Press, 1994.

[9] J. Serra, ed., *Image Analysis and Mathematical Morphology. II: Theoretical Advances.* London: Academic Press, 1988.

[10] H. J. A. M. Heijmans, "Connected morphological operators for binary images," Research Report PNA-R9708, CWI, 1997.

[11] C. Lantuéjoul and S. Beucher, "On the use of the geodesic metric in image analysis," *J. Microscopy,* vol. 121, pp. 29–49, 1980.

[12] C. Lantuéjoul and F. Maisonneuve, "Geodesic methods in quantitative image analysis," *Pattern Recognition,* vol. 17, pp. 177–187, 1984.

[13] E. Breen and R. Jones, "An attribute-based approach to mathematical morphology," in *Mathematical Morphology and its Application to Image and Signal Processing* (P. Maragos, R. W. Schafer, and M. A. Butt, eds.), pp. 41–48, Boston: Kluwer Academic Publishers, 1996.