Sample Statistic & Population Parameter

$E[X] = \frac{\sum X_i}{n} = \sum_{i=1}^{n} f(X_i)X_i = \mu$, where
E[X] = Rationally Expected value of X or the mean of X.
$f(X_i)$ = probability of X_i in functional form.
If we assume $f(X_i) = \frac{1}{n}$ for $\forall i$ (all i), then $\sum f(X_i)X_i = \frac{\sum X_i}{n}$
μ = population (true) mean of X.

$$\begin{aligned}\sigma_X^2 &= E[X-\mu]^2 = E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2 \quad (\because E[X] = \mu \quad \textit{from above})\end{aligned}$$, where

E = [.] is an Expected value of [.], which, in case of ***a constant***, is always ***constant*** (i.e. E[μ] = μ), b/c Expected value of a constant is still a constant.

Therefore, variance in another notation where there are k-number of sample groups with n-number of elements in each sample group would be

$\sigma^2 = \sum_i^k \sum_j^n (X_{ij} - \overline{X})^2 = \sum_i^k \sum_j^n (X_{ij}{}^2) - \frac{\sum_i^k \sum_j^n \overline{X}^2}{k \times n}$, which exactly concurs with the above notation of $\sigma^2 = E[X-\mu]^2 = E[X^2] - \mu^2$ with only the intermediate procedure omitted. This entails $\sigma_\varepsilon^2 = E[\varepsilon^2] - (E[\varepsilon])^2 = E[\varepsilon^2] - 0 = E[\varepsilon^2], \quad where \quad \varepsilon \sim iid.$

Sampling Distribution

$\mu = \mu_{\overline{X}} = E[\overline{X}]$ = Mean of the means

$\sigma_{\bar{x}}^2 = E[(\bar{x}_i - \mu)^2] = \frac{\sum (\bar{x}_i - \mu)^2 f(x_i)}{n} = \sigma^2 f(x_i) = \frac{\sigma^2}{n}$ = Variance of the means

$\sigma_{\bar{x}} = \sqrt{E[(\bar{x}_i - \mu)^2]} = \sqrt{\frac{\sum (\bar{x}_i - \mu)^2 f(x_i)}{n}} = \frac{\sigma}{\sqrt{n}}$ = stdev of the means or standard error,

where $f(x_i) = \frac{1}{n} \quad \forall\, i$

If $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$, then $z = \frac{(\overline{X} - \mu)}{\sigma / \sqrt{n}} = \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \sim N(0,1)$, where

$pdf(X \sim ND) = \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}}$ or $f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(X-\mu)^2}{2\sigma^2}}$, where $-\infty < X < \infty$

Sampling Distribution & Population Parameters

$$E[\overline{X}] = E\left[\frac{1}{n}\sum X_i\right] = \frac{1}{n}E\left[\sum X_i\right] = \frac{1}{n}\sum E\left[X_i\right] = \frac{1}{n}\sum \mu = \frac{1}{n}n\mu = \mu$$

$$E[s^2] = E\left[\frac{1}{n-1}\sum (X_i - \overline{X})^2\right] = \frac{1}{n-1}\sum E\left[(X_i - \overline{X})^2\right] = \frac{1}{n-1}\sum E[X_i^2 - 2X_i\overline{X} + \overline{X}^2]$$

$$= \frac{1}{n-1}\left\{\sum E[X_i^2] - 2E[\overline{X}\sum X_i] + \sum E[\overline{X}^2]\right\} = \frac{1}{n-1}\left\{\sum(\sigma^2 + \mu^2) - 2nE[\overline{X}^2] + \sum\left(\frac{\sigma^2}{n} + \mu^2\right)\right\}$$

$$= \frac{1}{n-1}\left\{n\sigma^2 + n\mu^2 - 2n\frac{\sigma^2}{n} - 2n\mu^2 + n\frac{\sigma^2}{n} + n\mu^2\right\} = \frac{1}{n-1}\left\{n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2\right\}$$

$$= \frac{1}{n-1}\left\{n\sigma^2 - \sigma^2\right\} = \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2$$