Linear Programming

1. **Assumptions & Concepts:** As opposed to optimizing production of only one single output, which would be attained invariably at a point where MR=MC or MR=0 or MC=0 on curvilinear TR and TC curves, linear programming is used when the firm produces more than one output under more than one constraints. It also assumes that
   i) The objective function as well as the constraints are linear and can be represented by straight lines.
   ii) Inputs and output prices are constant.
   iii) Production technology is constant returns to scale. (MC = AC & constant)
   iv) Output prices are constant. → Unit profit is constant & profit function is linear.
   v) The limited technologically fixed input combinations result not in smooth isoquants, but made up of straight-line segments.

Since firms face a number of constraints, and the objective function as well as constraints are often linear over the relevant range of operation, linear programming is quite useful.

2. Profit Maximization

a. Choice Variables: Assume two goods X & Y, where $A\pi_X = AR_X - AC_X = \$30$ & $A\pi_Y = AR_Y - AC_Y = \$40$, then our objective function is to maximize π, where $\pi = 30X+40Y$.

b. Constraints: Assume three inputs, A, B & C, where each of these inputs is required to produce X & Y only constrained by their availability as follows.

$1A(X)+1A(Y) < 7A$

$.5B(X)+.5B(Y) < 5B$

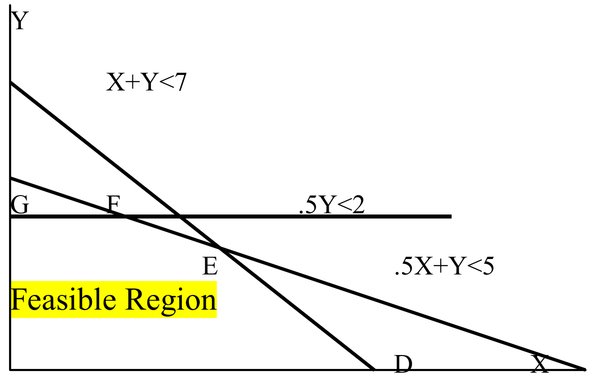
$.5C(Y) < 2C$

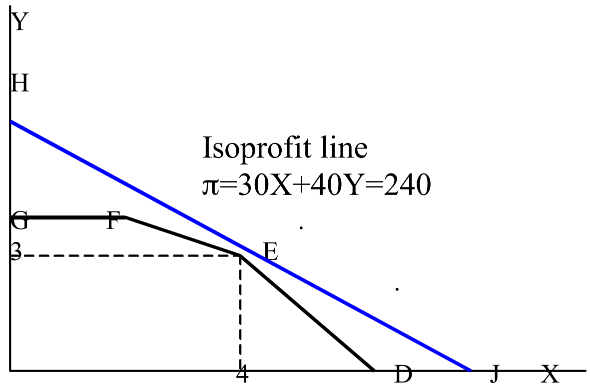
	Inputs required	per output	Inputs available
Inputs	X	Y	Total
A	1	1	7
B	.5	1	5
C	0	.5	2

Then, Objective function: Max $\pi = 30X+40Y$

subject to
$X+Y < 7$
$.5X+Y < 5$
$.5Y < 2$ (Slack Variable)
$X > 0, Y > 0$ (Non-negativity Constraint)

c. Graphic solution (Extreme Points and Simplex Method)
   i) Graph the constraints and find the feasible region.
   ii) Graph the isoprofit lines starting from the LCM of the π_X & π_Y.
   iii) Find the tangency point between the isoprofit line and the feasible region. Usually this can be done by checking the extreme points (corners) only (Simplex Method).





d. Algebraic Solution

i) Determine relevant extreme points on the feasible region, which are D, E, and F (G is dismissed, because it is an inefficient point, $\because Y_G = Y_F$, but $X_G < X_F$.), and calculate X & Y values at these points ignoring inequality signs.

a.) Point D: Solving X+Y=7 for Y=0, X=7.

b.) Point E: X+Y=7 ⋯⋯ (1)

.5X+Y=5 ⋯⋯ (2)

Y=3 ⋯⋯⋯2* (2)−(1)

$\therefore$ Y=3, X= 4

c.) Point F: .5Y=2 → .5X+Y=5

.5X+4=5

$\therefore$ X= 2, Y=4

ii) Compare the profits at these points.

a.) Point D: $30(7)+$40(0)=$210

b.) Point E: $30(4)+$40(3)=$240 *

c.) Point F: $30(2)+$40(4)=$220

e. LINDO Program

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max 30x+40y
st
1x+1y<7
.5x+1y<5
.5y<2
end
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3. Cost Minimization

a. Choice (Decision) Variables and Constraints

1P(X)+2P(Y) >14P

1M(X)+1M(Y) > 10M

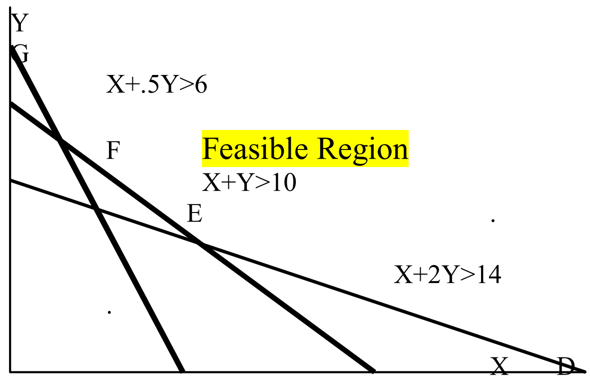
1V(X)+.5V(Y) > 6V

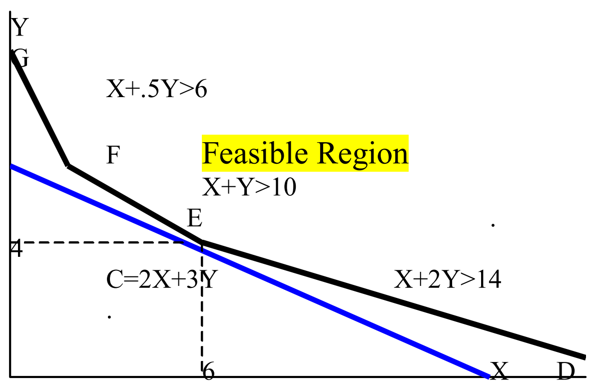
	Inputs required	per output	Inputs available
Inputs	X	Y	Total
P	1	2	≤14
M	1	1	≤10
V	1	.5	≤6

Then, Objective function: min C = 2X+3Y

subject to
X+2Y > 14
X+Y > 10
X+.5Y > 6
X > 0, Y > 0 (Non-negativity Constraint)

b. Graphic solution (Extreme Points and Simplex Method)





c. Algebraic Solution

i) Determine relevant extreme points on the feasible region, which are D, E, F and G, and calculate X & Y values at these points ignoring inequality signs.

a) Point D: Solving X+2Y=14 for Y=0, X=14.

b) Point E: X+2Y=14 ······ (1)
X+Y=10 ······ (2)
Y=4 ············ (1)-(2)
X=6
c) Point F: X+.5Y=6 ······ (3)
2X+Y=12··· 2*(3)
X+Y=10 ······ (4)
X=2 ········· 2*(3)-(4)
Y=8
d) Point F: Solving X+.5Y=6 for X=0, Y=12.

ii) Compare the profits at these points.
a) Point D: $2(14)+$3(0)=$28
b) Point E: $2(6)+$3(4)=$24 *
c) Point F: $2(2)+$3(8)=$28
d) Point G: $2(0)+$3(12)=$36

d. LINDO Program

```
min 2x+3y
st
1x+2y>14
1x+1y>10
1X+.5y>6
end
```

4. Dual Problem and Shadow Prices

a. Every linear programming (primal) problem has a corresponding or symmetric (dual) problem. The solutions to a dual problem are shadow prices, which give the change in the value of the objective function per unit change in each constraint in the primal problem. Shadow prices thus provide the imputed value or marginal valuation of each input. If a particular input is not fully employed, its shadow price is zero, because increasing the input would leave profits unchanged. A firm should increase the use of the input as long as marginal value or shadow price of the input > the cost of hiring the input.

b. Shadow prices are used i) for planning & strategic decisions; ii) to correctly price the output of each division that is the input to another division, in order to maximize the profit of the entire firm; iii) by governments to appropriately price some government services, for which there is no market; iv) for planning in developing countries where the market system often does not function properly.

c. Dual of Profit Maximization

i) Primal: Max π=30X+40Y s.t. X+Y<7, .5X+Y<5, .5Y<2, X>0, Y>0
Dual: minimize TC=$7V_A+5V_B+2V_C$
subject to X: $V_A+.5V_B \geq 30$
Y: $V_A+V_B+.5V_C \geq 40$
$V_A, V_B, V_C \geq 0$

The dual objective function is minimize V_A, V_B, and V_C (the imputed values or shadow prices of inputs A, B, and C respectively) subject to $V_A+.5V_B \geq 30$, $V_A+V_B+.5V_C \geq 40$, V_A, V_B, $.5V_C \geq 0$ (The sum of the shadow prices of each input used to produce 1 unit of X &/or Y must be ≥ the profit contribution of 1 unit of X &/or Y.).

ii) Solution: $V_A+V_B=40$ ········· (1)

$V_A+.5V_B=30$ ······ (2)

$V_A=20$, $V_B=20$ & $V_C=0$ as C is a slack variable.

This means that increasing A &/or B by 1 unit would increase the profit by \$20. Therefore, the firm should be willing to pay ≤ \$20 for 1 additional unit of A &/or B. Then, TC=7(20)+5(20)+2(0)=\$240.

d. Dual of Cost Minimization

i) Primal: min TC=2X+3Y s.t. x+2Y>14, X+Y>10, X+.5Y>6

Dual: Max $\pi=14V_P+10V_M+6V_V$

Subject to X: $V_P+V_M+V_V < 2$

Y: $2V_P+V_M+.5V_V < 3$

T: 3P+2M+1.5V

V_P, V_M, $V_V > 0$

ii) Solution: V is a slack variable. Therefore, $V_V=0$.

$V_P+V_M+V_V < 2$ ········· (1)

$2V_P+V_M+.5V_V < 3$······ (2)

$V_P=1$, $V_M=1$

Then, π=14(\$1)+10(\$1)+6(\$0)=\$24. If the profit contribution resulting from increasing the P and M constraints by 1 unit > V_P & V_M, the profit will increase.