ECON 343
Lecture 7 : Unit Roots and Co-integration

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Outline – Lecture 7

1. Testing for random walks
    1. Unit Root test
    2. Eviews example

2. Co-integrated time series
    1. Co-integration
    2. Examples of possible cointegrating relationships in finance
    3. Error Correction Models (ECM)

Reference: Pindyck and Rubinfeld, Ch. 15

Recap

- Many time series in economics and finance are not stationary but contain deterministic/stochastic trends
  - Series correlated with their past values (time dependent)
  - OLS results may not be valid
  - New models and tools needed
- If series are stationary: we can use **ARMA, ARIMA** and other linear stochastic specifications for forecasting
- If series *homogeneous non-stationary*: first difference will be stationary
- Important to have a *robust test* to determine if series is homogeneous non-stationary, in order to operate first (or more) differencing to induce stationarity (beyond graphical evaluation)

1. Testing for random walk

- **Question: Do economic variables such as GDP, employment and interest rates tend to revert back to some long-run trend following a shock, or do they follow random walks?**
  - **If random walk: spurious results in regression**
  - **De-trending will not help, but first-differencing will**
- **If a variable like GDP or stock index follow a random walk, then the effects of a temporary shock will not dissipate after several years, and might be permanent**
- **Random Walk general equation:**

$$y_t = \gamma^T y_{t-T} + \gamma \varepsilon_{t-1} + \gamma^2 \varepsilon_{t-2} + \ldots + \gamma^T \varepsilon_{t-T} + \varepsilon_t$$

Random Walk

- **Random Walk general equation:**

$$y_t = \gamma^T y_{t-T} + \gamma\varepsilon_{t-1} + \gamma^2\varepsilon_{t-2} + \ldots + \gamma^T\varepsilon_{t-T} + \varepsilon_t$$

- **Three cases:**

1. **Stationary case, shocks gradually die away**

$$\gamma \langle 1 \Rightarrow \gamma^T \rightarrow 0 \quad as \quad T \rightarrow \infty$$

2. **Shocks persist in the system and never die away**

   this is the UNIT ROOT case

$$\gamma = 1 \Rightarrow \gamma^T = 1 \quad \forall T$$

3. **Explosive case: shocks become more influential with time**

$$\gamma \rangle 1 \Rightarrow \gamma^3 \rangle \gamma^2 \rangle \gamma \quad etc.$$

Testing for Random Walks

Consider the stationary AR(1) process

$$y_t = \gamma y_{t-1} + \epsilon_t, \;\; \epsilon_t \sim WN(0, \sigma^2) \quad |\gamma| < 1. \tag{1}$$

Not that we have specified $-1 < \gamma < 1$ (commonly written $|\gamma| < 1$). If $|\gamma| = 1$ we have a unit root (we usually focus on the case $\gamma = +1$ in which case y_t is a <u>random walk</u>).

Subtract y_{t-1} from both sides of (1):

$$\Delta y_t \equiv y_t - y_{t-1} = \gamma^* y_{t-1} + \epsilon_t, \;\; \gamma^* = \gamma - 1. \tag{2}$$

Note that $-1 < \gamma < 1 \;\Rightarrow\; -1 < 1 + \gamma^* < 1 \;\Rightarrow -2 < \gamma^* < 0.$

Unit roots

Testing for a unit root is therefore equivalent to testing

$$H_0: \ \gamma^* = 0 \ \text{ unit root } \rightarrow (\gamma = 1)$$
$$\text{vs } H_1: \ \gamma^* < 0$$

This is most easily carried out using a one-sided t-test in (2), but the problem is that y_t is nonstationary under H_0, and so the usual limiting (normal) distribution doesn't apply for the t-ratio.

However, the appropriate distribution is the Dickey-Fuller distribution, for which critical values are tabulated. The test statistic is

$$DF = \frac{\hat{\gamma}^*}{se(\hat{\gamma}^*)},$$

where $se(\cdot)$ denotes the estimated standard error.

Unit roots

$$\Delta y_t \equiv y_t - y_{t-1} = \gamma^* y_{t-1} + \epsilon_t, \quad \gamma^* = \gamma - 1.$$

$H_0: \ \gamma^* = 0$ unit root
vs $H_1: \ \gamma^* < 0$

$$DF = \frac{\hat{\gamma}^*}{se(\hat{\gamma}^*)},$$

where $se(\cdot)$ denotes the estimated standard error.

Note that this is a <u>one-sided</u> test – we are looking for significantly <u>negative</u> values of DF in order to reject H_0.

Let $\overline{DF}$ denote the critical value (where $\overline{DF} < 0$):

Decision rule: if $DF \leq \overline{DF}$ reject H_0 in favour of H_1

if $DF > \overline{DF}$ do not reject H_0 (y_t is nonstationary)

More tests

1. *Random walk with drift:* often it is appropriate to include an intercept in the model, so that (1) becomes

$$y_t = \mu + \gamma y_{t-1} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2).$$

If $\gamma = 1$ then $\Delta y_t = \mu + \epsilon_t$ (random walk with drift) and changes in y_t are equal to a constant μ plus a random component ϵ_t. The test regression becomes

$$\Delta y_t = \mu + \gamma^* y_{t-1} + \epsilon_t,$$

and the same testing procedure applies (but with different critical values).

More tests

2. *Random walk with drift and trend:* sometimes a linear trend is appropriate, in which case (1) becomes

$$y_t = \mu + \beta t + \gamma y_{t-1} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2).$$

If $\gamma = 1$ then $\Delta y_t = \mu + \beta t + \epsilon_t$ (<u>random walk with drift and trend</u>) and changes in y_t are equal to a linear trend $\mu + \beta t$ plus a random component ϵ_t. The test regression becomes

$$\Delta y_t = \mu + \beta t + \gamma^* y_{t-1} + \epsilon_t,$$

and the same testing procedure applies (but with different critical values).

More tests

3. If an AR(p) model is more appropriate than an AR(1) we can augment the test regression with $p-1$ lags of Δy_t, yielding:

$$\begin{aligned}
\Delta y_t &= \gamma^* y_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta y_{t-i} + \epsilon_t, \\
\Delta y_t &= \mu + \gamma^* y_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta y_{t-i} + \epsilon_t, \\
\Delta y_t &= \mu + \beta t + \gamma^* y_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta y_{t-i} + \epsilon_t.
\end{aligned}$$

This is known as the augmented Dickey-Fuller, or ADF, test, and can also have the effect of removing autocorrelation from the residuals (which is desirable because the critical values depend on ϵ_t being white noise).

Eviews example

- Do commodity prices follow random walks?
  - Historical gold and oil prices, since 1870
  - DF test: random walk is also called I(1): integrated of order 1

Unrestricted model:

$$P_t - P_{t-1} = \alpha + \beta t + (\gamma - 1)P_{t-1} + \lambda \Delta P_{t-1} + \varepsilon_t$$

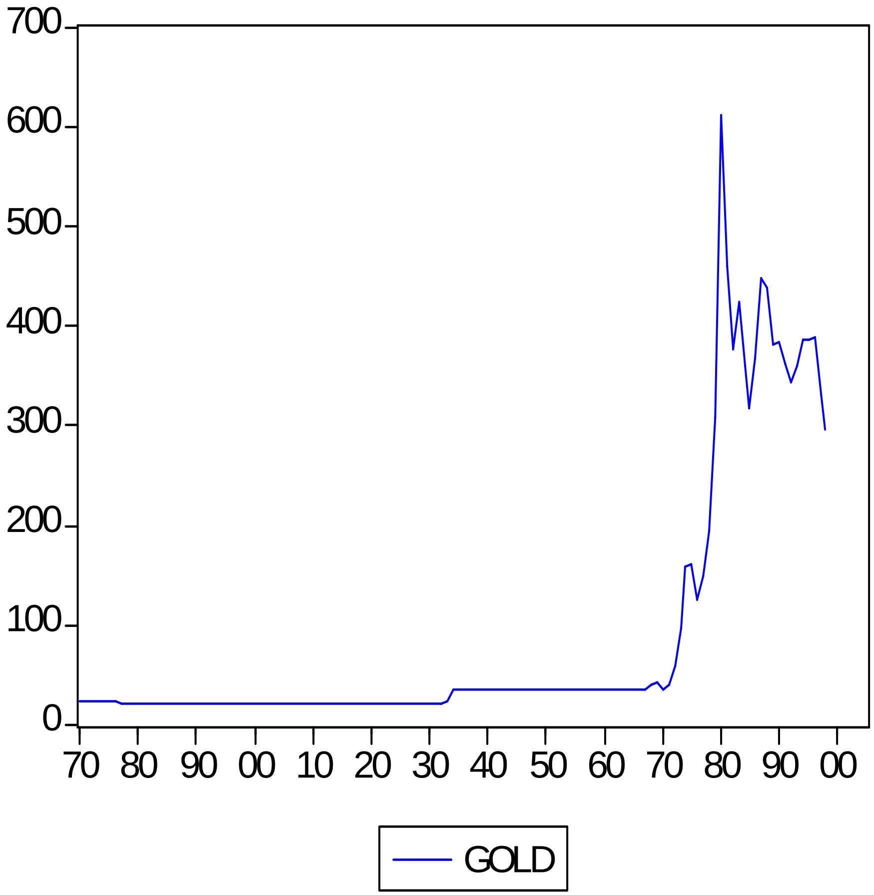
Restricted model:

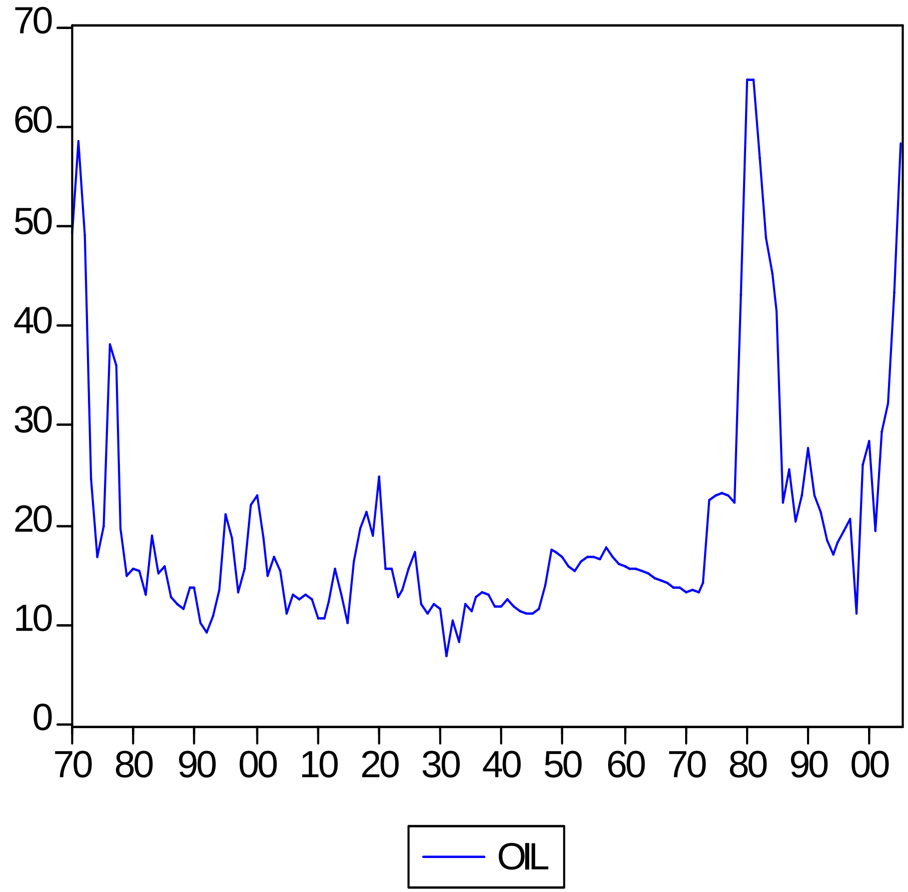
$$P_t - P_{t-1} = \alpha + \lambda \Delta P_{t-1} + \varepsilon_t$$

If prices follow random walk with trend, then:

$$P_t = \alpha + \beta t + \gamma P_{t-1} + \varepsilon_t$$

Eviews example





Eviews example: oil

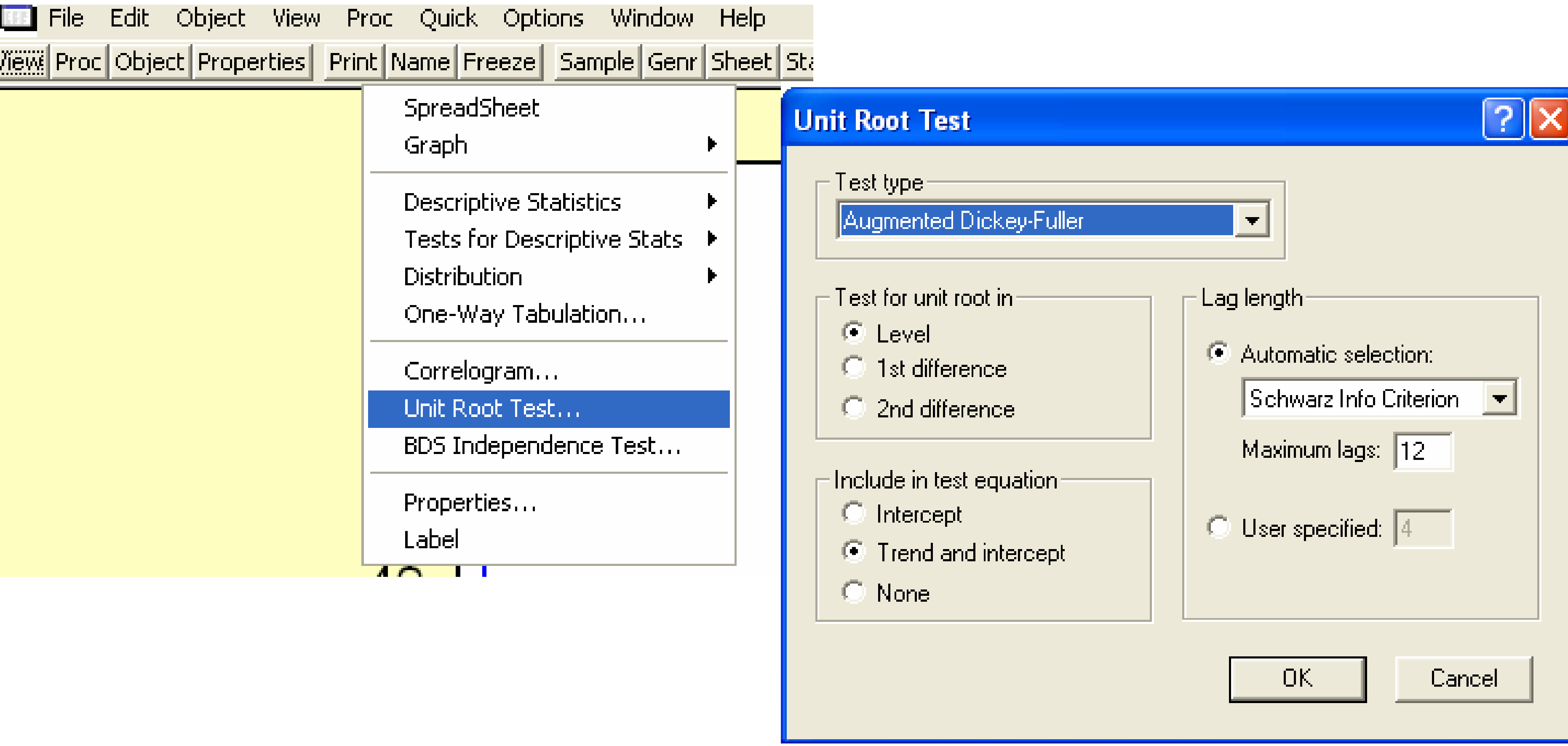
Level correlogram

Lag	AC	PAC	Q-Stat	Prob
1	0.808	0.808	90.799	0.000
2	0.584	-0.199	138.60	0.000
3	0.443	0.112	166.27	0.000
4	0.384	0.103	187.22	0.000
5	0.370	0.084	206.89	0.000
6	0.285	-0.195	218.64	0.000
7	0.168	-0.049	222.75	0.000
8	0.112	0.096	224.60	0.000
9	0.074	-0.088	225.41	0.000
10	0.047	-0.023	225.73	0.000
11	0.013	-0.001	225.76	0.000
12	0.010	0.119	225.78	0.000
13	0.007	-0.089	225.78	0.000
14	0.000	0.004	225.78	0.000
15	0.007	0.075	225.79	0.000
16	0.002	-0.040	225.79	0.000
17	0.028	0.066	225.91	0.000
18	0.074	0.062	226.77	0.000
19	0.111	0.069	228.75	0.000
20	0.151	0.033	232.43	0.000
21	0.152	-0.045	236.22	0.000
22	0.188	0.185	242.05	0.000
23	0.237	0.014	251.37	0.000
24	0.238	-0.089	260.88	0.000
25	0.189	-0.094	266.89	0.000
26	0.097	-0.061	268.50	0.000
27	0.051	0.035	268.96	0.000
28	0.061	-0.011	269.60	0.000

First difference correlogram

Lag	AC	PAC	Q-Stat	Prob
1	0.049	0.049	0.3093	0.578
2	-0.263	-0.266	9.4569	0.009
3	0.049	0.084	9.7774	0.021
4	-0.144	-0.243	12.561	0.014
5	-0.119	-0.056	14.489	0.013
6	0.141	0.046	17.199	0.009
7	0.233	0.216	24.685	0.001
8	0.089	0.110	25.792	0.001
9	-0.076	-0.009	26.607	0.002
10	-0.072	-0.015	27.330	0.002
11	-0.110	-0.074	29.050	0.002
12	-0.068	-0.026	29.715	0.003
13	0.055	-0.035	30.160	0.004
14	0.056	-0.049	30.613	0.006
15	0.020	-0.037	30.673	0.010
16	0.068	0.076	31.360	0.012
17	-0.087	-0.063	32.497	0.013
18	-0.125	-0.025	34.876	0.010
19	-0.040	-0.070	35.119	0.014
20	-0.012	-0.024	35.140	0.019
21	0.011	-0.036	35.158	0.027
22	0.010	-0.058	35.174	0.037
23	-0.023	-0.079	35.256	0.049
24	-0.029	-0.029	35.393	0.063
25	-0.015	0.029	35.429	0.081
26	-0.006	0.022	35.435	0.103
27	-0.000	0.017	35.435	0.128
28	-0.000	-0.015	35.435	0.158
29	-0.003	-0.004	35.437	0.191

Eviews example



Eviews example: oil

Null Hypothesis: OIL has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 1 (Automatic based on SIC, MAXLAG=12)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-4.727892	0.0010
Test critical values:	1% level	-4.027959	
	5% level	-3.443704	
	10% level	-3.146604	

*MacKinnon (1996) one-sided p-values.

Reject H0

Eviews example: Gold

Null Hypothesis: GOLD has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic based on SIC, MAXLAG=12)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-2.217764	0.4754
Test critical values:	1% level	-4.031309	
	5% level	-3.445308	
	10% level	-3.147545	

*MacKinnon (1996) one-sided p-values.

Do not Reject H0

2. Co-integration

Suppose we have 2 independent random walks:

$$y_t = y_{t-1} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2),$$

$$x_t = x_{t-1} + u_t, \quad u_t \sim WN(0, \omega^2),$$

where $E(\epsilon_t u_s) = 0$ for all t and s.

Suppose we estimate the model $y_t = \beta_1 + \beta_2 x_t + v_t$, obtaining the fitted equation

$$y_t = b_1 + b_2 x_t + e_t.$$

Co-integration

$$y_t = b_1 + b_2 x_t + e_t.$$

We would probably expect:

(i) $t_2 = b_2 / se(b_2)$ to be 'small,' thereby not rejecting $H_0 : \beta_2 = 0$;

(ii) R^2 to be low,

because y_t and x_t are completely independent of each other.

Co-integration

But, because both y_t and x_t are I(1) (random walks), we have what is known as a spurious regression, and we find that:

(i) $t_2 \rightarrow \infty$ as $T \rightarrow \infty$ i.e. the larger the sample, the higher the probability of *rejecting* $H_0 : \beta_2 = 0$;

(ii) R^2 will be 'acceptable,' suggesting a reasonable fit.

In the spurious regression, e_t will also be I(1), but suppose we found a situation where it was I(0). What does this imply about the relationship between y_t and x_t?

Co-integration

$$y_t = b_1 + b_2 x_t + e_t.$$

In the spurious regression, e_t will also be I(1), but suppose we found a situation where it was I(0). What does this imply about the relationship between y_t and x_t?

It suggests that the linear combination $y_t - b_1 - b_2 x_t$ of the two I(1) series is stationary, and in this case y_t and x_t are said to be cointegrated.

Many time series in economics have been found to be I(1), and so cointegration is often important for economic theory to make much sense!!

Co-integration test

1. Test for the orders of integration of the variables y_t and x_t.

2. If both variables are integrated of the same order, estimate (by OLS) the equation

$$y_t = \beta_1 + \beta_2 x_t + v_t,$$

obtaining the fitted equation

$$y_t = b_1 + b_2 x_t + e_t.$$

3. To test for cointegration, apply the DF/ADF test to e_t. If $e_t \sim I(1)$ there is no cointegration, but if $e_t \sim I(0)$, there is evidence of cointegration. *Note:* a different set of critical values is required for the DF/ADF tests of cointegration.

Examples of co-integrating relationships in finance

- Financial theory suggests where two or more variables would be expected to hold some long run relationship with one another
  - Spot and future prices for a given commodity or asset
  - Ratio of relative prices and an exchange rate
  - Equity prices and dividends
- In all three cases, market forces arising from no-arbitrage conditions suggest that there should be an equilibrium relationship between the series concerned

Eviews example

- **Is the LEVANT price index co-integrated with oil prices? Is there a long run relationship between oil prices and stock market?**
  - **Daily data on LEVANT stock (Reuters) and oil prices (NYSE)**

1. Test for the orders of integration of the variables y_t and x_t.
2. If both variables are integrated of the same order, estimate (by OLS) the equation

$$y_t = \beta_1 + \beta_2 x_t + v_t,$$

   obtaining the fitted equation

$$y_t = b_1 + b_2 x_t + e_t.$$

3. To test for cointegration, apply the DF/ADF test to e_t. If $e_t \sim I(1)$ there is no cointegration, but if $e_t \sim I(0)$, there is evidence of cointegration. *Note:* a different set of critical values is required for the DF/ADF tests of cointegration. See Greene, p656 for references.

1. Unit root tests

Null Hypothesis: OIL has a unit root
Exogenous: Constant
Lag Length: 0 (Automatic based on SIC, MAXLAG=15)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-2.168435	0.2185
Test critical values:	1% level	-3.454626	
	5% level	-2.872121	
	10% level	-2.572482	

*MacKinnon (1996) one-sided p-values.

Null Hypothesis: LEV has a unit root
Exogenous: None
Lag Length: 0 (Automatic based on SIC, MAXLAG=15)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		1.962683	0.9883
Test critical values:	1% level	-2.574208	
	5% level	-1.942094	
	10% level	-1.615856	

*MacKinnon (1996) one-sided p-values.

2. Regression

Equation Estimation

Specification | Options

Equation specification

Dependent variable followed by list of regressors including ARMA and PDL terms, OR an explicit equation like Y=c(1)+c(2)*X.

```
log(lev) c log(oil)
```

Equation: UNTITLED   Workfile: TEST1\Untitled

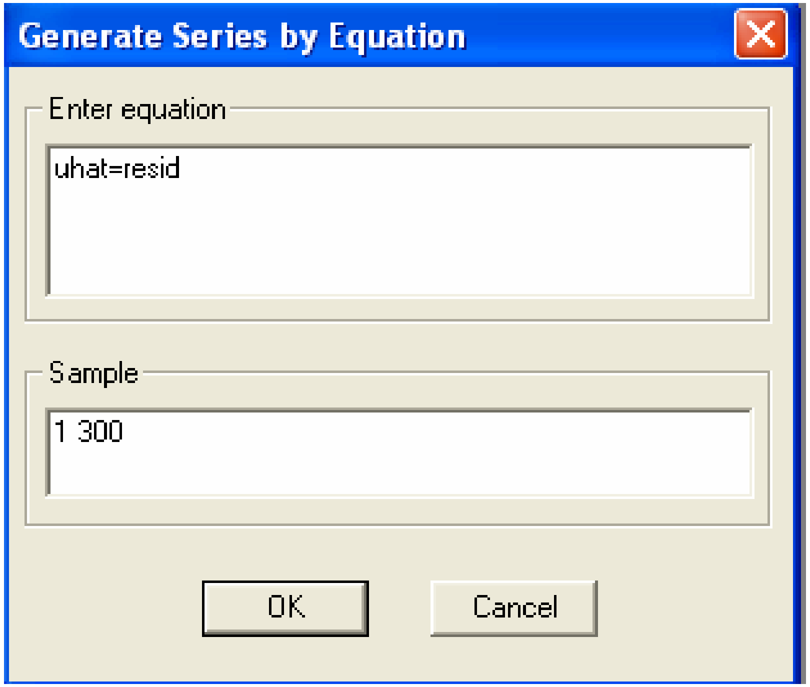
View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: LOG(LEV)
Method: Least Squares
Date: 03/29/06   Time: 15:25
Sample (adjusted): 1 252
Included observations: 252 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	1.867675	0.316484	5.901317	0.0000
LOG(OIL)	1.541675	0.078560	19.62423	0.0000

R-squared	0.606368	Mean dependent var	8.075906
Adjusted R-squared	0.604793	S.D. dependent var	0.228170
S.E. of regression	0.143440	Akaike info criterion	-1.037892
Sum squared resid	5.143775	Schwarz criterion	-1.009880
Log likelihood	132.7744	F-statistic	385.1105
Durbin-Watson stat	0.066351	Prob(F-statistic)	0.000000

3. Residual unit root



Null Hypothesis: UHAT has a unit root
Exogenous: None
Lag Length: 0 (Automatic based on SIC, MAXLAG=15)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-1.649951	0.0934
Test critical values:	1% level	-2.574208	
	5% level	-1.942094	
	10% level	-1.615856	

*MacKinnon (1996) one-sided p-values.

Weak rejection of H0 of residual: LEV index and OIL weakly co-integrated

Error Correction Model

- First-differencing a random walk series allows regressions
- but may results in loss of information about the long-run relationship between two variables
- Co-integration helps to describe the long-run, or equilibrium, relationship between the variables
- Their dynamic behavior can be described by an error correction model or ECM, of the form:

$$\Delta y_t = \lambda e_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + \sum_{i=1}^{q} \delta_i \Delta x_{t-i} + \epsilon_t,$$

where $\epsilon_t \sim WN(0, \sigma^2)$.

Error Correction Model

$$\Delta y_t = \lambda e_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + \sum_{i=1}^{q} \delta_i \Delta x_{t-i} + \epsilon_t,$$

where $\epsilon_t \sim WN(0, \sigma^2)$. Here y_t is responding not only to lagged changes in y and x, but also to last period's disequilibrium or error, e_{t-1}. We expect $\lambda < 0$:

if $e_{t-1} < 0 \Rightarrow y_{t-1} < b_1 + b_2 x_{t-1} \Rightarrow \Delta y_t$ should be positive;

if $e_{t-1} > 0 \Rightarrow y_{t-1} > b_1 + b_2 x_{t-1} \Rightarrow \Delta y_t$ should be negative.