ECON 343
Lecture 6 : Properties of Stochastic Time Series

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Outline – Lecture 6

1. **Introduction to stochastic time series models**
   1. Random walks
   2. Stationary and Nonstationary time series
   3. Properties of stationary processes
2. **Characterizing time series: the autocorrelation function**
   1. The autocorrelation function
   2. Homogenous Nonstationary processes
   3. Stationarity and the Autocorrelation function
3. **Eviews applications**
   1. Interest rate: three-month US Treasury bill rate
   2. Comparative per capita real GDP, Yemen and South Korea

Reference: Pindyck and Rubinfeld, Ch. 15

1. Introduction

- Previous lecture: deterministic models of time trends
  - Time plays the only role in determining evolution
- This lecture: stochastic (random) Data Generating Process
  - Time-series models that provide a description of the random nature of the stochastic process in the data
- No causality involved (like in regression), but determining the influence of this randomness
- Important assumption:
  - Each value in the series $y_1, y_2, \ldots, y_T$ is drawn randomly from a probability distribution
  - Equivalently: y_T drawn from a set of jointly distributed random variables
  - Aim: specify the relevant probability distribution in order to determine the probability if occurrence of future outcomes

Random walks

- Simplest example of stochastic time series
- Each successive change in y_t is drawn independently from a probability distribution with 0 mean
- Model:

$$y_t = y_{t-1} + \varepsilon_t \qquad E(\varepsilon_t) = 0 \qquad E(\varepsilon_t \varepsilon_s) = 0 \quad for \quad t \neq s$$

- Forecast of this process

$$\hat{y}_{T+1} = E(y_{T+1} | y_T, \ldots, y_1)$$

- But

$$y_{T+1} = y_T + \varepsilon_{T+1} \quad independen\,t \quad of \quad y_{T-1}, \ldots, y_1$$

- Thus the forecast one period ahead is simply:

$$\hat{y}_{T+1} = y_T + E(\varepsilon_{T+1}) = y_T$$

Random walks

- Forecast two periods ahead:

$$\hat{y}_{T+2} = E(y_{T+2} \mid y_T, \ldots, y_1) = E(y_{T+1} + \varepsilon_{T+2})$$

$$= E(y_T + \varepsilon_{T+1} + \varepsilon_{T+2}) = y_T$$

- Similarly, forecast *l* periods ahead is also y_T
- Although the forecast ahead will be the same, the variance of the forecast error will grow as *l* becomes larger
- One period forecast error

$$e_1 = y_{T+1} - \hat{y}_{T+1} = y_T + \varepsilon_{T+1} - y_T = \varepsilon_{T+1} \qquad \text{var}: E(\varepsilon_{T+1}^2) = \sigma_\varepsilon^2$$

- Two period forecast error

$$e_2 = y_{T+2} - \hat{y}_{T+2} = y_T + \varepsilon_{T+1} + \varepsilon_{T+2} - y_T = \varepsilon_{T+1} + \varepsilon_{T+2}$$

$$\text{var}: E[(\varepsilon_{T+1} + \varepsilon_{T+2})^2] = E(\varepsilon_{T+1}^2) + E(\varepsilon_{T+2}^2) + 2E(\varepsilon_{T+1}\varepsilon_{T+2}) = 2\sigma_\varepsilon^2$$

Random walks

- Forecast l periods ahead, error variance is $\mathrm{var} = l\sigma_{\varepsilon}^{2}$
- The standard error of forecast increases with the square root of l
- <u>Confidence intervals</u> for forecasts become wider as the forecast horizon increases

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- <u>Random walk with drift:</u> embody trend in forecasts

$$y_t = y_{t-1} + d + \varepsilon_t$$

- One period forecast error

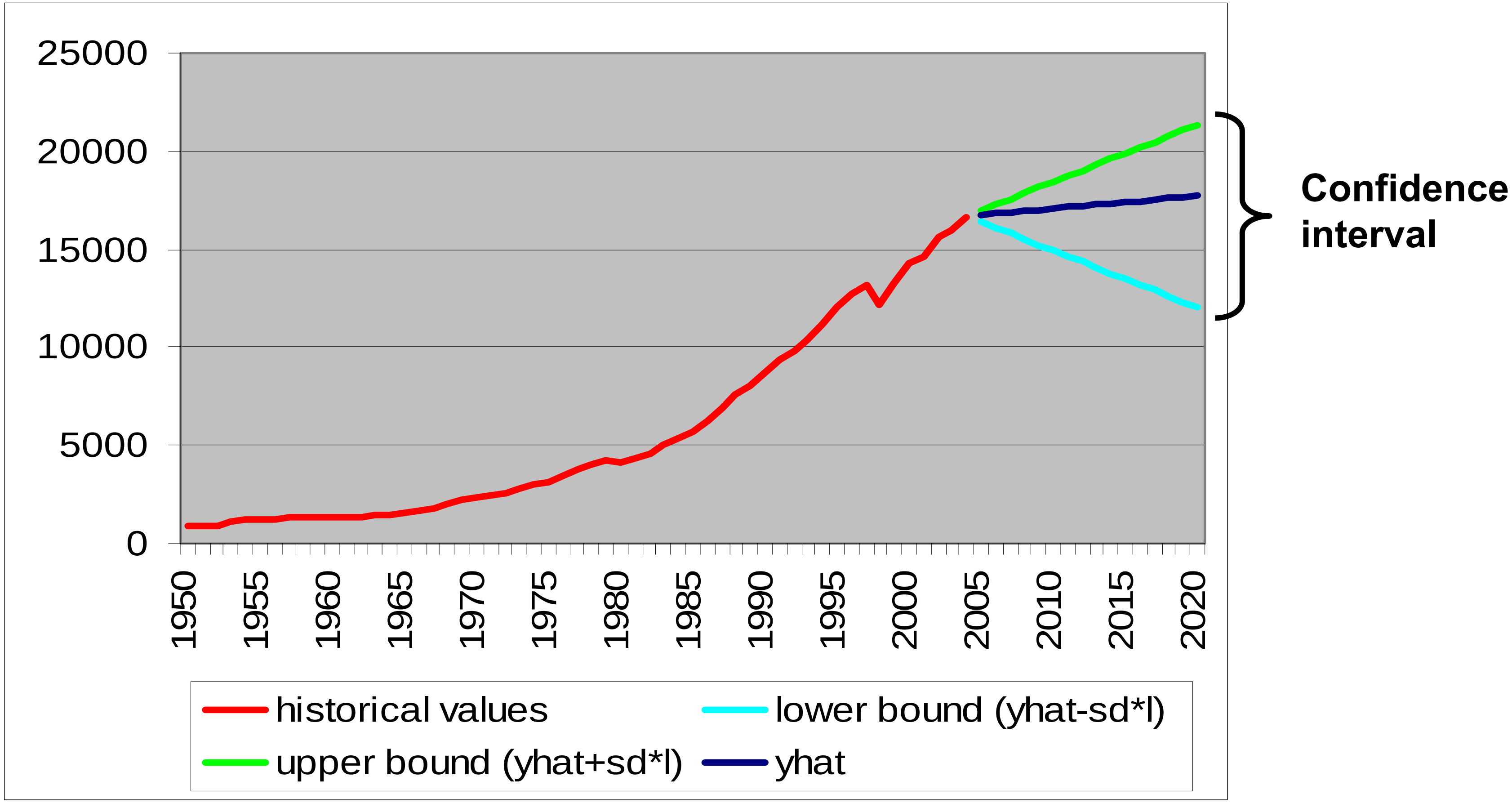
$$\hat{y}_{T+1} = E(y_{T+1} | y_T, \ldots, y_1) = y_T + d$$

- l period ahead: $\hat{y}_{T+1} = y_T + ld$
- Standard error of forecast will be the same as before

$$e_1 = y_{T+1} - \hat{y}_{T+1} = y_T + d + \varepsilon_{T+1} - y_T - d = \varepsilon_{T+1}$$

Random walks with drift

- **Example: South Korea real GDP per capita evolution**



Stationarity and Nonstationarity

- Definition: stochastic process generating a time series is said to be stationary if it is invariant with respect to time
- Regression models: structural relationship described by the equation is invariant over time – it is very important to test this assumption
- Stationarity: the probability of a given fluctuation in the process from the mean level is assumed to be the same at any point in time
- Many time series may not be stationary (evolution of GDP), but there are techniques to transform nonstationary processes into stationary ones

Properties of stationary processes

- **Definitions:**
  - $y_1, \ldots, y_T$ generated by a set of jointly distributed random variables – <u>joint probability distribution</u> function:

$$p(y_1, \ldots, y_T)$$

  - A future observation y_{T+1} generated by a <u>conditional probability distribution</u> function:

$$p(y_{T+1} | y_1, \ldots, y_T)$$

- **Theorem:**
  - A series y_t is stationary if and only if:

$$p(y_t, \ldots, y_{t+k}) = p(y_{t+m}, \ldots, y_{t+k+m})$$

and

$$p(y_t) = p(y_{t+m})$$

for any *t*, *k* and *m*.

Properties of stationary processes

- If the series yt is stationary, then:
  - The <u>mean</u> of the series must be stationary:

$$E(y_t) = E(y_{t+m}) \longleftrightarrow \mu_y = E(y_t)$$

  - The <u>variance</u> of the series must be stationary:

$$E[(y_t - \mu_y)^2] = E[(y_{t+m} - \mu_y)^2] \qquad \sigma_y^2 = E[(y_t - \mu_y)^2]$$

  - The <u>covariance</u> of the series must be stationary:

$$Cov(y_t, y_{t+k}) = Cov(y_{t+m}, y_{t+m+k})$$

$$\gamma_k = Cov(y_t, y_{t+k}) = E[(y_t - \mu_y)(y_{t+k} - \mu_y)]$$

- If a stochastic process is stationary, then an estimate of the mean can be obtained from the <u>sample mean</u> of the series:

$$\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t \qquad \text{var} : \hat{\sigma}_y^2 = \frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})$$

2. The Autocorrelation function

- **Definition:**
  - **The autocorrelation function tells us how much correlation there is between neighboring data points in the time series**
  - **<u>Autocorrelation with lag k:</u>**

$$\rho_k = \frac{E[(y_t - \mu_y)(y_{t+k} - \mu_y)]}{\sqrt{E[(y_t - \mu_y)^2]E[(y_{t+k} - \mu_y)^2]}} = \frac{Cov(y_t, y_{t+k})}{\sigma_{y_t}, \sigma_{y_{t+k}}}$$

- **For a stationary time series, the autocorrelation function becomes**

$$\rho_k = \frac{E[(y_t - \mu_y)(y_{t+k} - \mu_y)]}{\sigma_y^2} = \frac{\gamma_k}{\gamma_0}$$

The Autocorrelation function

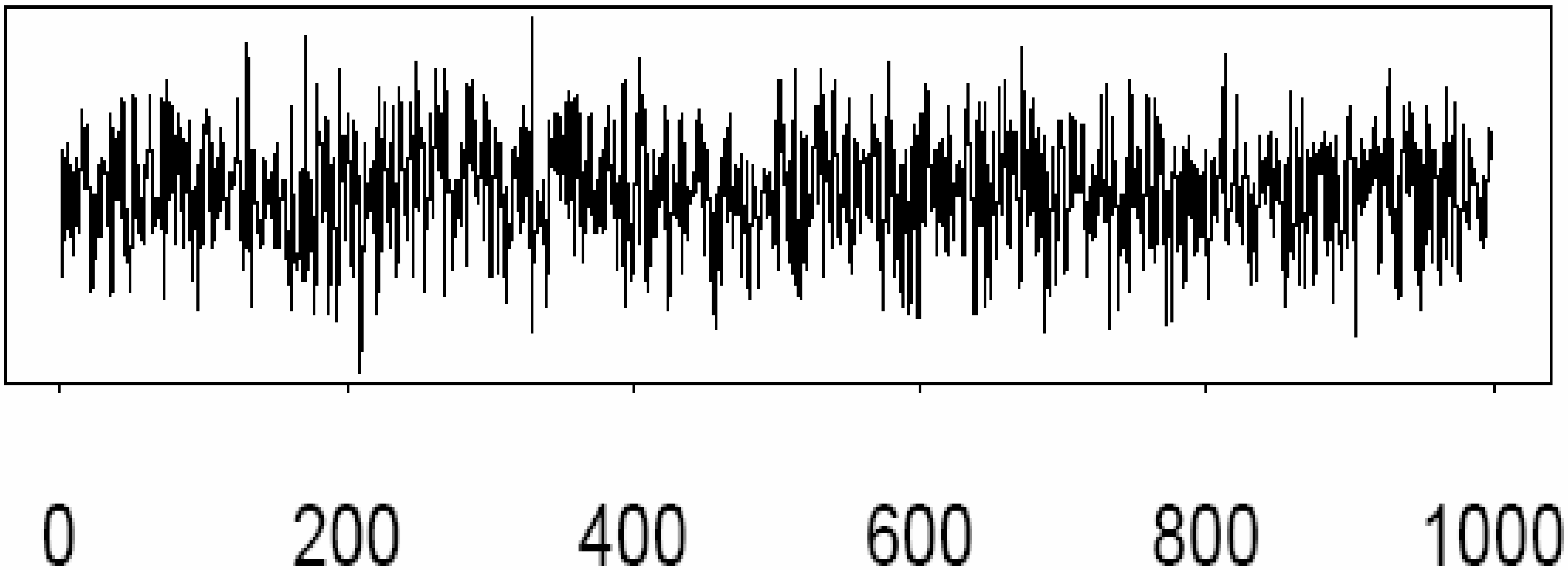
- The <u>sample autocorrelation function</u> is an estimate of the theoretical autocorrelation function:

$$\hat{\rho}_k = \frac{\sum_{t=1}^{T-k}(y_t - \bar{y})(y_{t+k} - \bar{y})}{\sum_{t=1}^{T}(y_t - \bar{y})^2}$$

- A <u>White Noise</u> process is one where the autocorrelation function is 0 for all k>0
  - Equation: $y_t = \varepsilon_t \quad \varepsilon_t \to iid$
  - Autocorrelation function: $\rho_0 = 1, \rho_k = 0, \forall k > 0$
  - Forecast: $\hat{y}_{T+l} = 0$

White Noise: illustration

White noise: iid $N(0,1)$ data



Statistical tests

- Test whether a particular value of the autocorrelation function is equal to zero
  - Bartlett test: Sample Autocorrelation Coefficient AC > $1/\sqrt{T}$, then 95% sure that true AC is not zero (T=sample size)
  - Because a white noise time series has AC normally distributed with mean 0 and standard deviation $1/\sqrt{T}$
- Test the joint hypothesis that all ACs are zero:
  - Q-statistic distributed as Chi square with K degrees of freedom
  - If Q(calculated)>Q(critical value,5% confidence) then we are 95% sure that all true ACs are not all zero

$$Q = T\sum_{k=1}^{K} \hat{\rho}_k^2$$

Homogeneous Nonstationary processes

- **Definitions:**
  - **If a nonstationary series is differenced one or more times, the resulting series will be stationary**
  - **Order of homogeneity: the number of times that the original series must be differenced before a stationary series is obtained**
  - **If y_t is first-order homogeneous nonstationary, then w_t is stationary:**

$$w_t = y_t - y_{t-1} = \Delta y_t$$

  - ***Random walk* process is first-order homogeneous stationary:**

$$y_t = y_{t-1} + \varepsilon_t$$

y_t is nonstationary (variance is infinite)

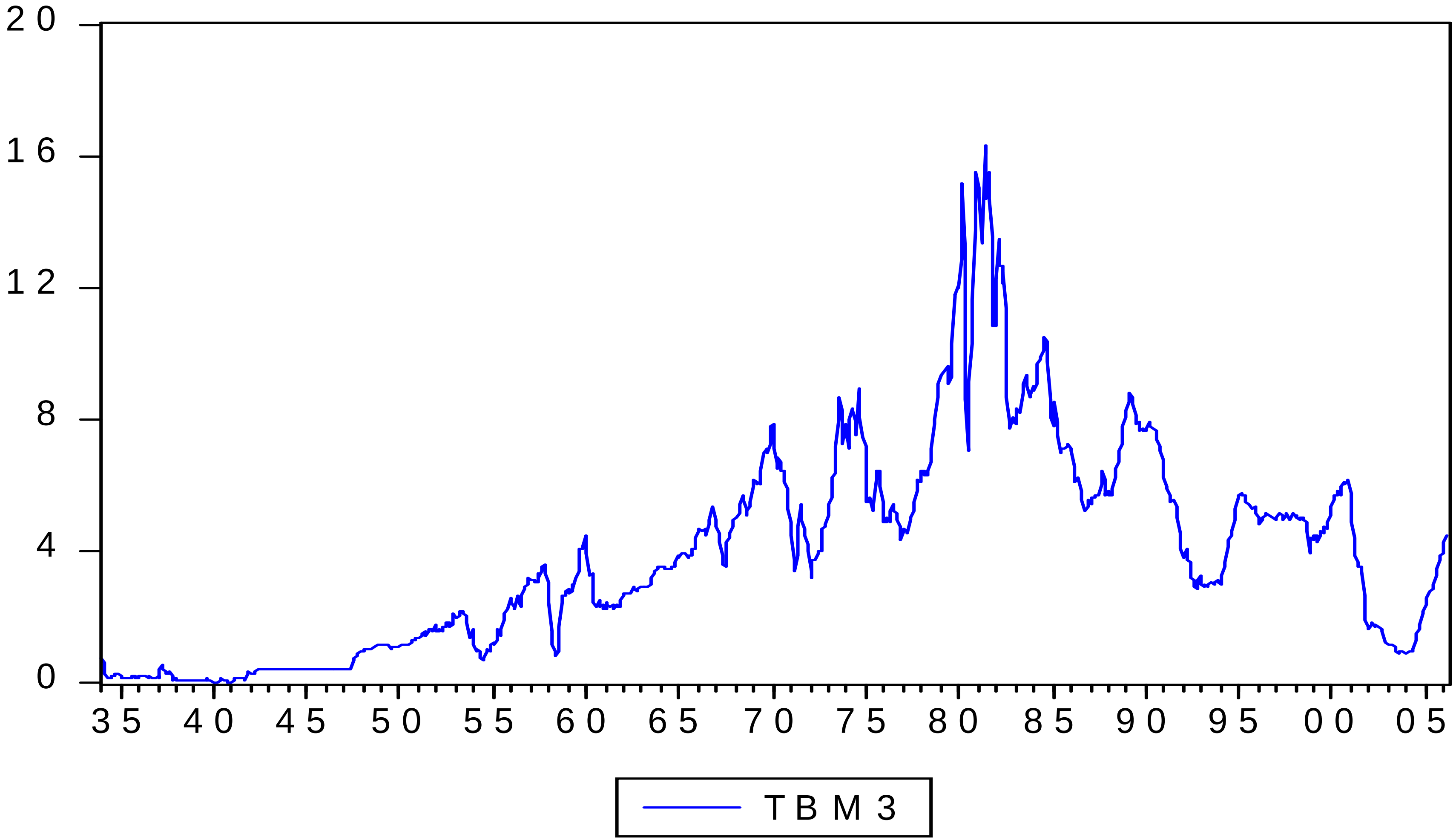
But first difference is stationary (white noise):

$$w_t = y_t - y_{t-1} = \Delta y_t = \varepsilon_t$$

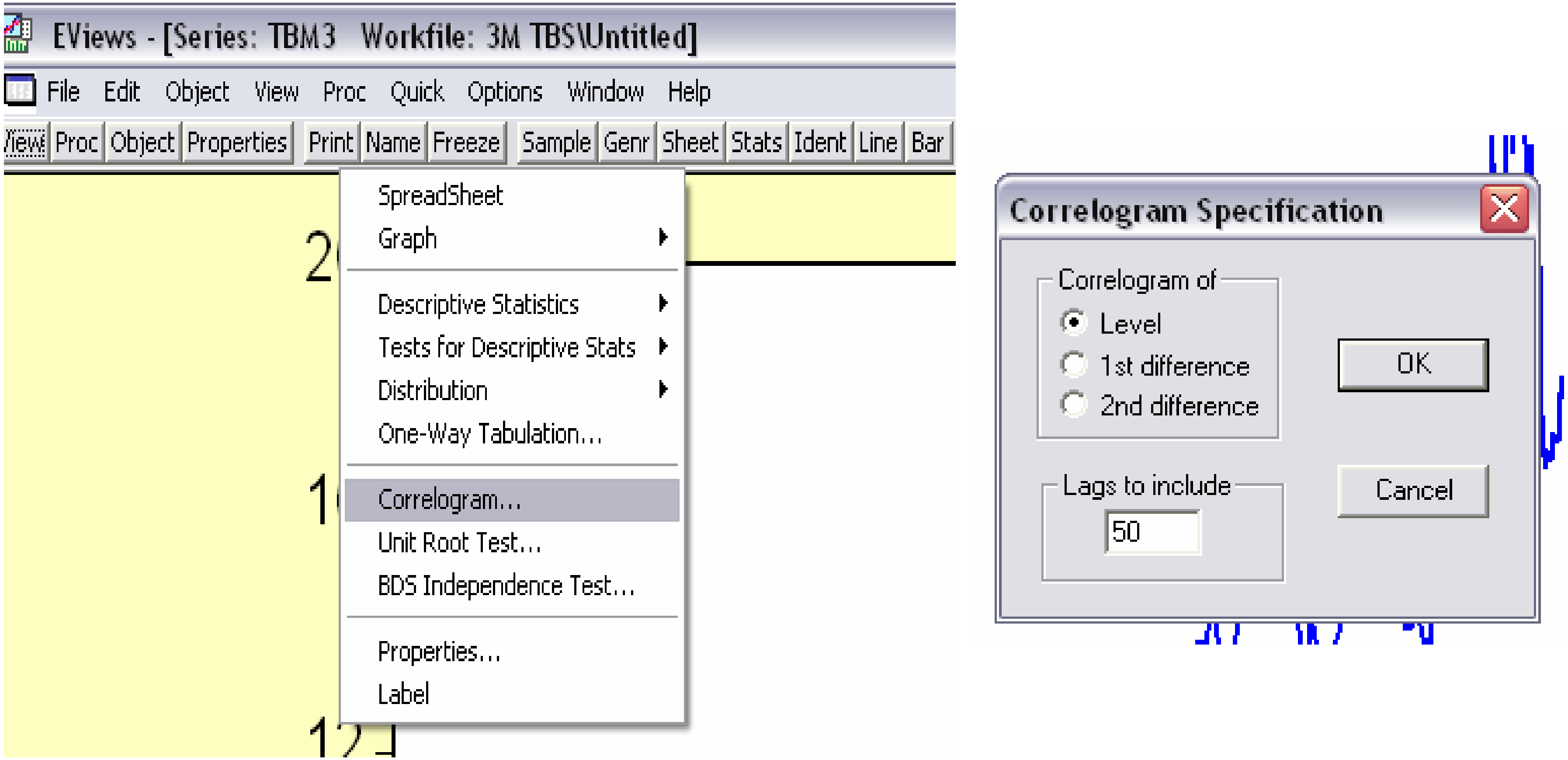
Stationarity and the Autocorrelation Function

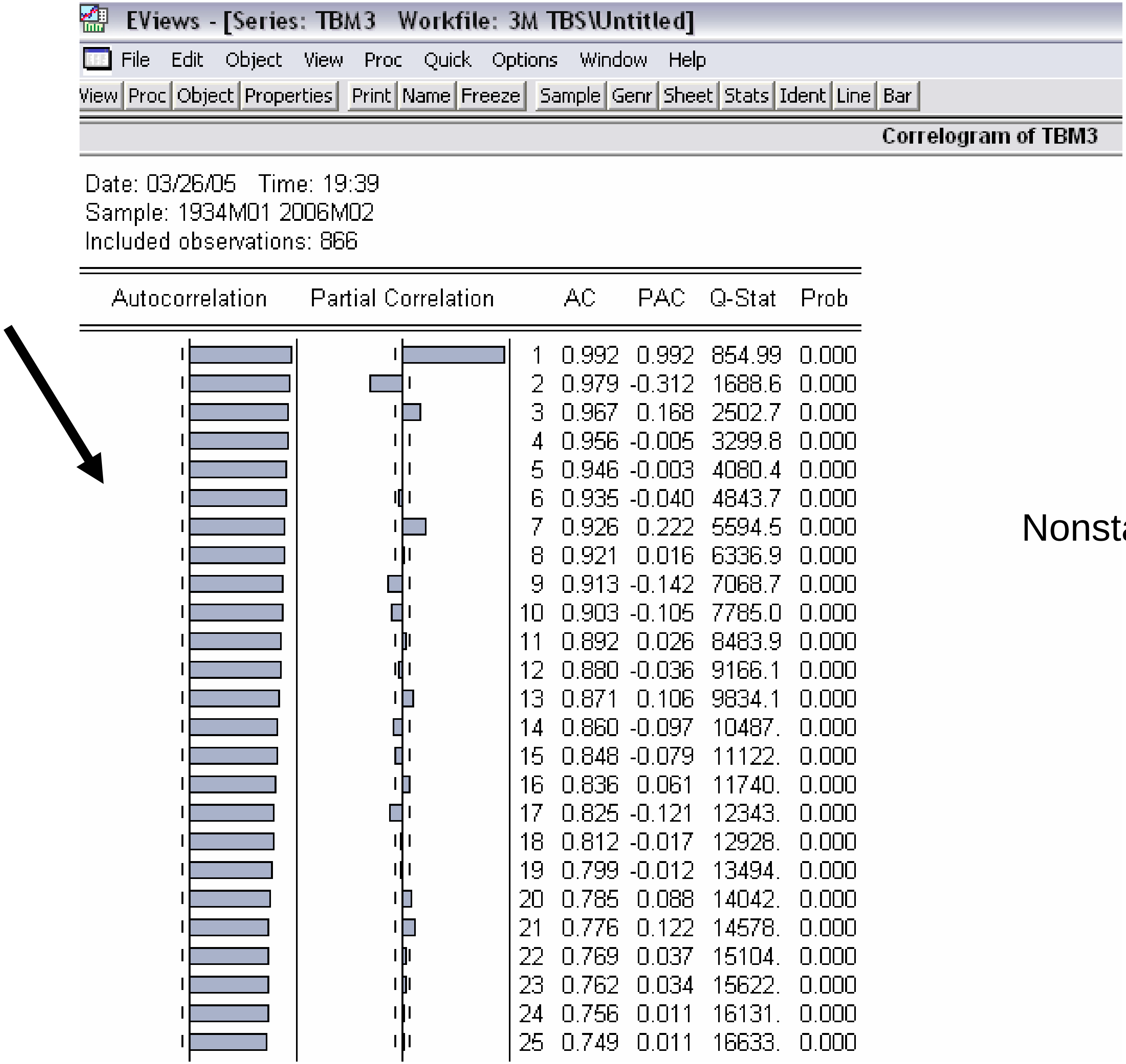
- Most series in economics and finance (GDP, stock market indices) are nonstationary (i.e. time dependent)
- Forecasting: difference the series one or two times, build model for new series, forecast then integrate (undifference) the model and its forecasts to get original results
- Key aspects:
  - Is the series stationary? Look at correlogram (plot of the autocorrelation function): if series stationary correlogram should decrease as the number of lags k increases
  - How many times should a homogeneous nonstationary series be differenced to get a stationary series? Look at correlogram of the first, second… difference functions, and the speed at which this correlogram drops

Eviews applications:
Monthly US 3-month T-Bill rates



Graphical tests of stationarity

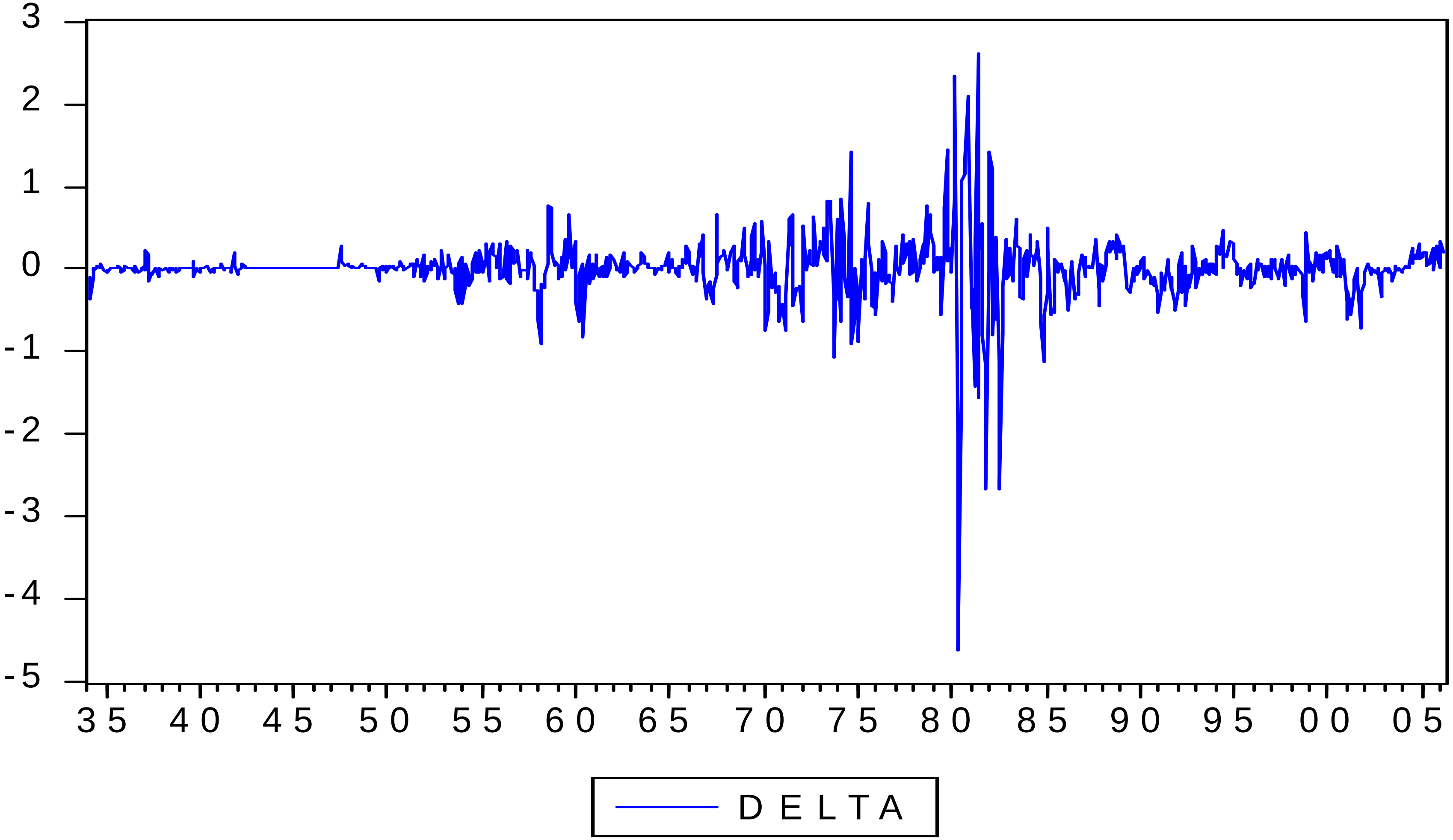




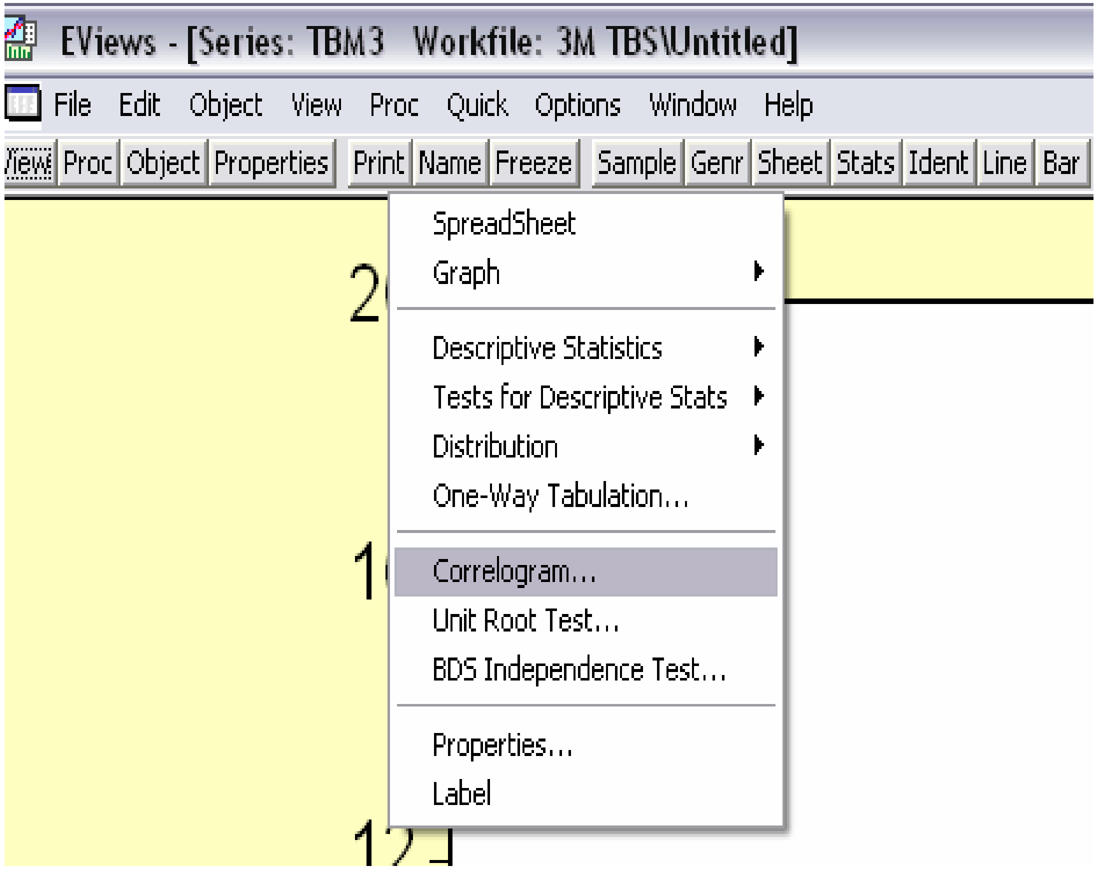
Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.992	0.992	854.99	0.000
		2	0.979	-0.312	1688.6	0.000
		3	0.967	0.168	2502.7	0.000
		4	0.956	-0.005	3299.8	0.000
		5	0.946	-0.003	4080.4	0.000
		6	0.935	-0.040	4843.7	0.000
		7	0.926	0.222	5594.5	0.000
		8	0.921	0.016	6336.9	0.000
		9	0.913	-0.142	7068.7	0.000
		10	0.903	-0.105	7785.0	0.000
		11	0.892	0.026	8483.9	0.000
		12	0.880	-0.036	9166.1	0.000
		13	0.871	0.106	9834.1	0.000
		14	0.860	-0.097	10487.	0.000
		15	0.848	-0.079	11122.	0.000
		16	0.836	0.061	11740.	0.000
		17	0.825	-0.121	12343.	0.000
		18	0.812	-0.017	12928.	0.000
		19	0.799	-0.012	13494.	0.000
		20	0.785	0.088	14042.	0.000
		21	0.776	0.122	14578.	0.000
		22	0.769	0.037	15104.	0.000
		23	0.762	0.034	15622.	0.000
		24	0.756	0.011	16131.	0.000
		25	0.749	0.011	16633.	0.000

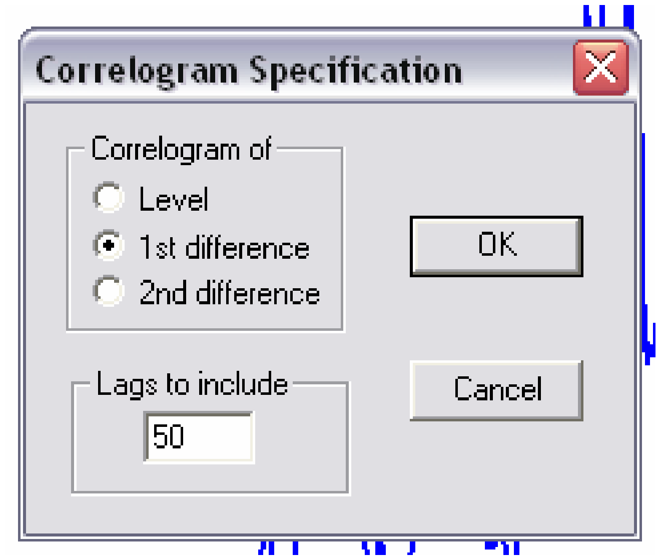
Nonstationary

First difference



Graphical tests of stationarity





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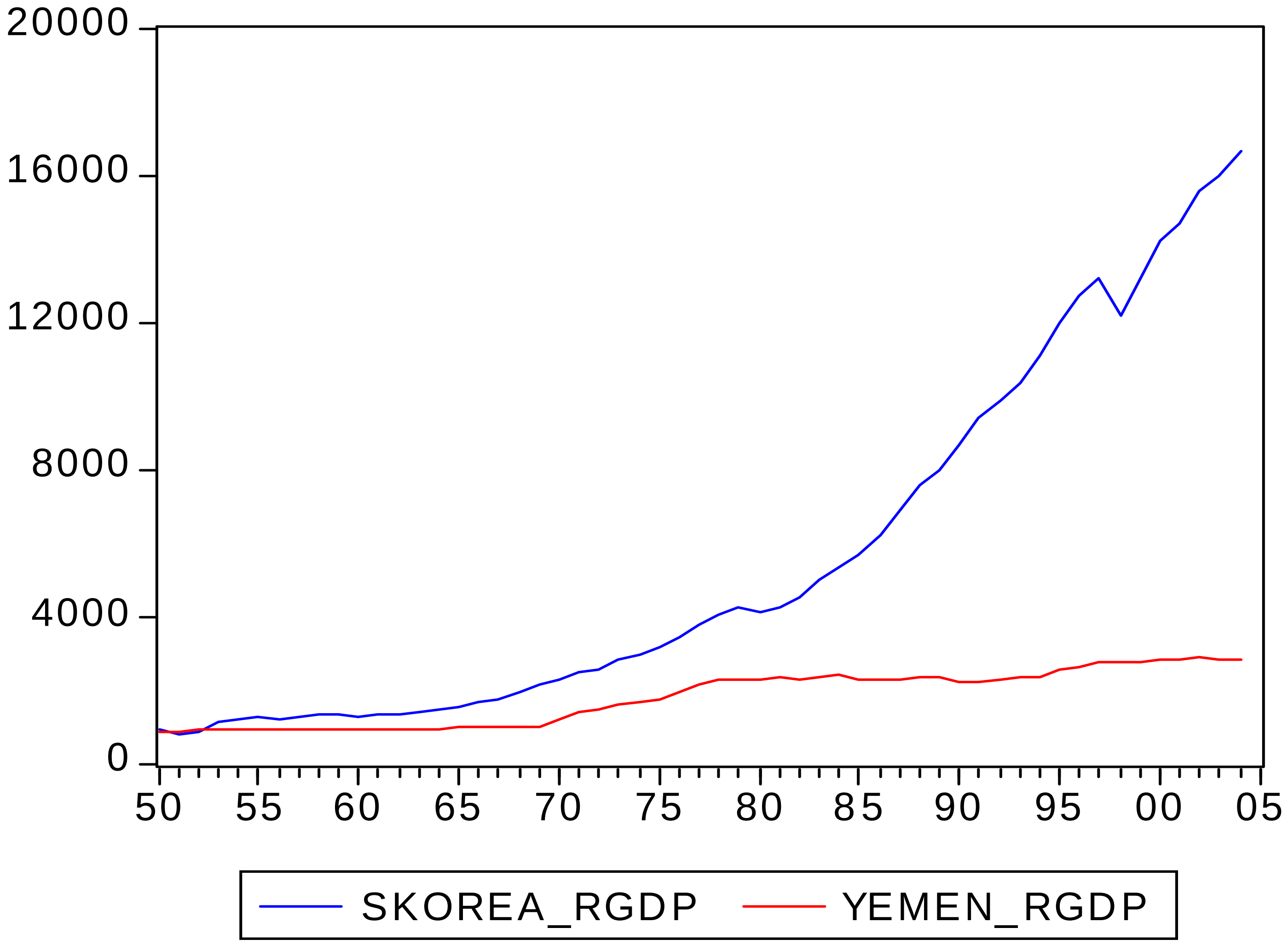
Correlogram of D(DELTA)

Date: 03/26/05 Time: 19:42
Sample: 1934M01 2006M02
Included observations: 864

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.191	-0.191	31.719	0.000
		2	-0.303	-0.353	111.66	0.000
		3	-0.053	-0.236	114.13	0.000
		4	0.004	-0.232	114.15	0.000
		5	0.212	0.061	153.25	0.000
		6	-0.176	-0.212	180.35	0.000
		7	-0.189	-0.283	211.67	0.000
		8	0.107	-0.206	221.67	0.000
		9	0.188	-0.041	252.75	0.000
		10	-0.025	-0.130	253.28	0.000
		11	-0.006	0.059	253.32	0.000
		12	-0.188	-0.202	284.38	0.000
		13	0.069	-0.113	288.52	0.000
		14	0.198	0.006	323.21	0.000
		15	-0.206	-0.179	360.70	0.000
		16	0.018	-0.048	360.98	0.000
		17	0.025	-0.037	361.53	0.000
		18	0.107	0.064	371.74	0.000
		19	0.055	0.053	374.42	0.000
		20	-0.190	0.002	406.47	0.000
		21	-0.053	-0.030	408.94	0.000
		22	0.077	-0.046	414.27	0.000
		23	0.034	-0.041	415.29	0.000
		24	-0.020	-0.024	415.63	0.000
		25	-0.069	-0.111	419.82	0.000

Comparative per capita real GDP, Yemen and South Korea



First difference correlogram, Yemen

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.500	0.500	14.292	0.000
		2	0.339	0.118	20.984	0.000
		3	0.254	0.061	24.811	0.000
		4	0.085	-0.117	25.252	0.000
		5	0.217	0.239	28.151	0.000
		6	0.139	-0.052	29.362	0.000
		7	-0.040	-0.200	29.466	0.000
		8	-0.094	-0.107	30.052	0.000
		9	-0.290	-0.215	35.690	0.000
		10	-0.258	-0.024	40.269	0.000
		11	-0.155	0.043	41.962	0.000
		12	-0.255	-0.132	46.659	0.000
		13	-0.242	-0.082	50.994	0.000
		14	-0.334	-0.134	59.428	0.000
		15	-0.255	0.124	64.454	0.000
		16	-0.155	-0.049	66.357	0.000
		17	-0.105	0.011	67.267	0.000
		18	-0.085	-0.102	67.877	0.000
		19	-0.092	-0.022	68.601	0.000
		20	-0.074	0.013	69.085	0.000
		21	-0.070	-0.141	69.532	0.000
		22	-0.065	-0.136	69.930	0.000
		23	-0.047	-0.117	70.141	0.000
		24	-0.052	-0.040	70.416	0.000
		25	-0.041	-0.016	70.593	0.000

First difference correlogram, S. Korea

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.322	0.322	5.9183	0.015
		2	0.194	0.100	8.1009	0.017
		3	0.287	0.224	12.984	0.005
		4	0.318	0.193	19.118	0.001
		5	0.358	0.226	27.050	0.000
		6	0.205	-0.006	29.690	0.000
		7	0.113	-0.073	30.518	0.000
		8	0.191	0.018	32.919	0.000
		9	0.153	-0.047	34.500	0.000
		10	0.086	-0.067	35.013	0.000
		11	0.057	-0.037	35.244	0.000
		12	0.050	-0.011	35.422	0.000
		13	0.064	0.002	35.725	0.001
		14	0.054	0.021	35.943	0.001
		15	0.040	0.043	36.066	0.002
		16	0.035	0.025	36.162	0.003
		17	0.029	0.007	36.231	0.004
		18	0.041	0.018	36.370	0.006
		19	0.037	0.005	36.489	0.009
		20	0.048	0.021	36.696	0.013
		21	0.038	-0.002	36.827	0.018
		22	0.026	-0.010	36.890	0.024
		23	0.037	-0.001	37.027	0.032
		24	0.036	-0.005	37.161	0.042
		25	0.019	-0.020	37.197	0.055

Second difference correlogram, Yemen

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.347	-0.347	6.7532	0.009
		2	-0.063	-0.209	6.9837	0.030
		3	0.085	-0.017	7.4086	0.060
		4	-0.301	-0.338	12.792	0.012
		5	0.216	-0.021	15.625	0.008
		6	0.108	0.133	16.346	0.012
		7	-0.119	0.016	17.249	0.016
		8	0.126	0.068	18.282	0.019
		9	-0.220	-0.124	21.494	0.011
		10	-0.014	-0.091	21.507	0.018
		11	0.108	-0.051	22.319	0.022
		12	-0.121	-0.129	23.362	0.025
		13	0.109	-0.072	24.220	0.029
		14	-0.116	-0.174	25.228	0.032
		15	-0.109	-0.188	26.134	0.037
		16	0.046	-0.226	26.301	0.050
		17	0.030	-0.085	26.373	0.068
		18	0.003	-0.147	26.374	0.092
		19	-0.006	-0.209	26.377	0.120
		20	-0.000	-0.111	26.377	0.154

Second difference correlogram, S. Korea

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.431	-0.431	10.416	0.001
		2	-0.158	-0.422	11.843	0.003
		3	0.134	-0.216	12.895	0.005
		4	0.017	-0.108	12.913	0.012
		5	0.090	0.146	13.407	0.020
		6	0.016	0.292	13.422	0.037
		7	-0.009	0.384	13.428	0.062
		8	0.018	0.449	13.448	0.097
		9	0.009	0.518	13.454	0.143
		10	0.005	0.606	13.455	0.199