Ilia Tutberidze

On Four-Colour Theorem

Let us have any planar map with arbitrary nodes with indices[1] $K \geq 3$. Let us consider any node of such kind (Fig.1). Let us encircle certain space R around the node and consider this space R as an area[2]. In this case we see that the index of nodes A, B, C, D, E is K=3.

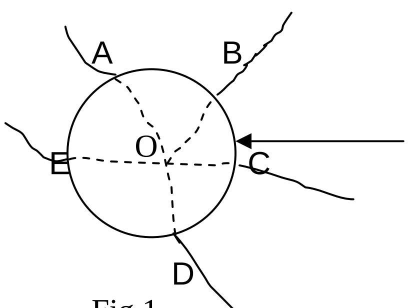


Fig.1

Let us have a map and it has the nodes with the index 3. Let us assume that colouring of such map is possible in N colours. In this case we can colour above mentioned map in N colours in such manner that each space R_n, made around all nodes with $K \geq 3$, has its own colour from the mentioned N colours. Then, we can colour sector EOD in the colour of the area, adjoining to EOD by line ED. Analogously we can act with the other 4 sectors**.** In this case it is clear that the initial map will be coloured in N colours in such manner that adjacent areas will have different colours.

We assume, that there is a group of areas inside any not one-connected area on the map and every area of this group has a generic side[3] at least with one area from this group (Fig.2) (the outer contour of this group in the Figure is line SMOCPN). Let us choose

[1] K-index node-point wich has K "branches" (e.g. in Fig.1 the index of node O is 5, and in Fig.2 given points S, M, O, N, P, C are nodes.

[2] "State", represented in area-map.

[3] Side – separating boundary of two areas.

arbitrary points K and Y on the outline (which does not contain any nodes) of area A and join them with an arbitrary point X on line SMOCPN. Consider space KXY as an area. In this case any area in the Figure has a generic side with another one and closed lines have at least two nodes (on the contour of A area two nodes - K and Y, appear).

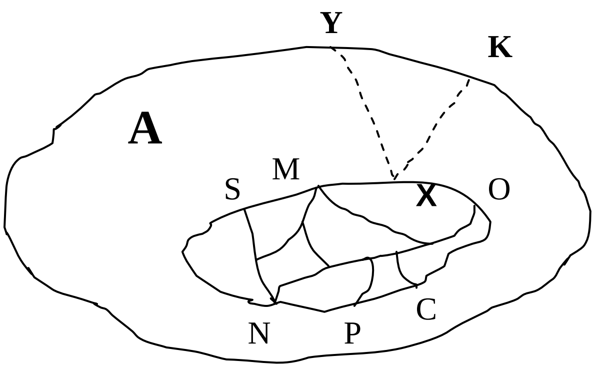


Fig. 2

Let the mentioned map in Fig.2 be coloured in N colours, area KXY has its own colour. Then, let us colour KXY with the same colour that the area, adjoining with KXY by sides KX and YX. It is easy to see that in this case primary map also will be coloured with N colours in such manner that every adjacent area will be coloured in different colours

Thus, it is shown, that in four-colour problem an arbitrary planar map may be transformed in such manner that the following cases take place.

1) We have only such nodes on the map, the index of which is K=3,
2) Each area represented on the map is 1-connected, and any closed line has at least two nodes.

Let us have a map, satisfying the above-mentioned two conditions (Fig.3).

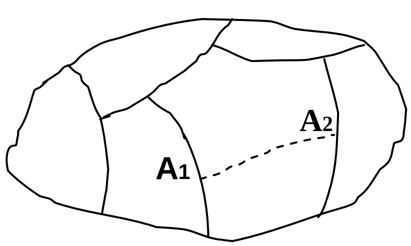


fig.3

It is clear, that by adding any side A_1A_2, the number of existing areas on the map increases by 1, the number of nodes – by 2. Accounting the fact that in case of only one area there are no nodes on the map, we have

$$P_n = 2(n-1) \qquad (1)$$

where P_n is the number of existed nodes on the map, n – the number of areas on the map.

Proceeding from the fact, that 3 sides on the map correspond to one node, and 2 nodes correspond to 1 side, then

$$L_n = 3/2\,P_n \qquad (2)$$

where L_n is the number of sides on the map.

Let us consider a group of areas such that every area of this group has at least one side with the all other areas and an arbitrary closed line contains at least two nodes. For simplicity we call such group a secondary communication group[4] (based on profit of the problem, we consider a secondary communication group, because as it was shown above, a case of primary group reduces to the case of a secondary group (Fig.2)). We say that two areas have communication if they have at least one generic side. It is clear that

$$S_z = 1/2\,Z(Z-1) \qquad (3)$$

where S_z is full number of communications existing in the communication group, Z is a number of areas in the communication group.

It is clear, that for communication group, which is made with Z number of areas, the following equality is true

$$L_z - L_{Limit} - \sum_{p=1}^{S_z}(Q_p(x,y)-1) = S_z \qquad (4)$$

[4] A Primary communication group – the group, where every area has at least one generic side with the other areas and the outer contour of mentioned group does not content eny nodes.

where L_z is full number of sides, L_{Limit} is a number of sides on the boundary of the communicational group (these sides do not make communications between areas of communicational group), $Q_p(x,y)$ is a number of generic sides of arbitrary areas X and Y of communication group.

Issue from (1), (2), (3) expressions, expression (4) has a form

$$1/2Z^2 - 7/2Z + 3 + L_{Limit} + \sum_{p=1}^{S_z}(Q_p(x,y)-1) = 0 \quad (5)$$

where, proceeding from the interpretation of the secondary group of communication

$$L_{Limit} \geq 2 \qquad (6)$$

It is easy to see, that when $L_{Limit} + \sum_{p=1}^{S_z}(Q_p(x,y)-1) \geq 4$, then expression (5) has no solution.

On the basis of above-mentioned, let us consider the following variants

1) $L_{Limit} = 2$ and $\sum_{p=1}^{S_z}(Q_p(x,y)-1) = 1$

In this case the solutions of eq. (5) are Z=3 (Fig.4) and Z=4 (Fig.5).

(As seen in Figs. 4 and 5, $Q_p(1,3)=2$).

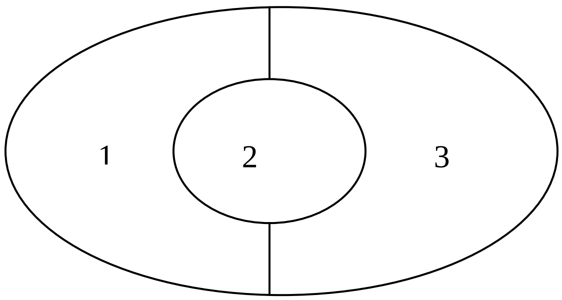


Fig.4

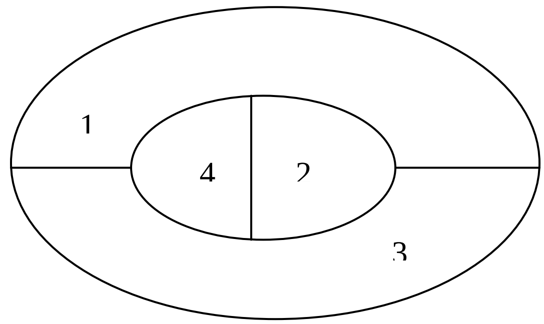


Fig.5

2) $L_{Limit} = 3$, $\sum_{p=1}^{S_z} (Q_p(x,y) - 1) = 0$.

In this case the solutions of eq. (5) are Z=3 (Fig.6) and Z=4 (Fig.7).

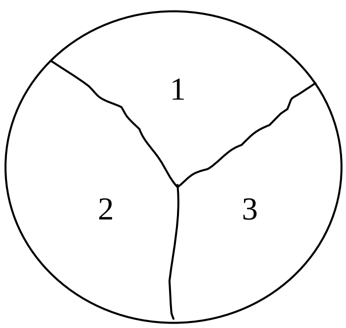


Fig.6

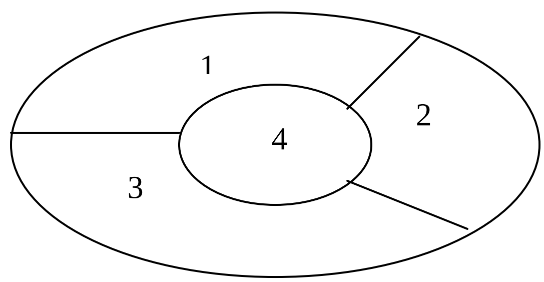
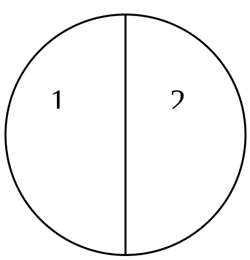


Fig.7

3) $L_{Limit} = 2$, $\sum_{p=1}^{S_z} (Q_p(x,y) - 1) = 0$.

In this case the solutions of equation (5) are Z=2 (Fig.8) and Z=5.

Fig.8



Let us consider the case, when Z=5, $L_{Limit} = 2$ and $\sum_{p=1}^{S_z}(Q_p(x,y)-1)=0$.

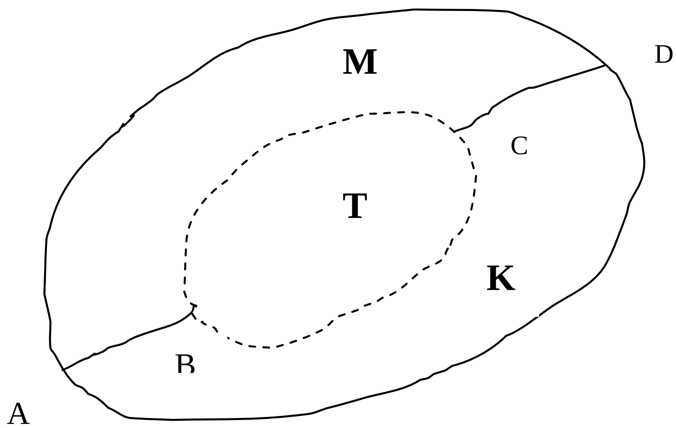


Fig.9

In Fig.9 four sides are represented. 2 sides are located on the boundary of communicational group (on the outer boundary we have two nodes – D and A) and 2 other sides are AB and CD. Assume that the other 8 sides (when Z=5, L_z=12) are “located” in T area, the boundary of which is noted by dotted line. It is evident, that areas T and M on the map are those parts of the communicational groups, the sides of which present boundary sides of given communication groups and sides AB and CD.But in this case the condition $\sum_{p=1}^{S_z}(Q_p(x,y)-1)=0$, is broken because sides AB and CD are two generic sides of the mentioned communication groups. Thus, the 3rd version for area Z=5 is invalid.

Hence, for arbitrary map the following is proved:
The maximal number of areas of communication group is 4.

Let we have any not coloured planar map, where we choose a group, consisted of 4 areas. Let us every area of this group has a generic side at least with one area from the same group (it is not necessary for this group to be a communication group of second order). It is evident that in this case four colours are enough for colouring of this group. Then we can choose the fifth area which should have a generic side with any area of the mentioned group. Clearly, it is not necessary to use a different colour for colouring of the 5^{th} area, as maximal number of areas of communication group is 4 (so maximum 3 areas of the initial group consisted with 4 areas, can create a communication group together with 5^{th} area). Now, we can add 6^{th} area to the group consisted of 5 areas. It is not necessary to use fifth colour for colouring 6 th area as maximal number of communication group is 4 (so maximum 3 areas of the initial group consisted with 5 areas, can create a communication group with 6^{th} area). We can continue such operation until the whole map coloured without using the fifth colour.