Limites Envolvendo a Seqüência de Euler

Leohnard Euler, 1707-1783, (27/08/01 11:50:50, arquivo: aueulog.doc

Usar o limite fundamental:

$\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ **ou** $\lim_{t\to 0}(1+t)^{1/t} = e$

Limites da Seqüência de Euler: $(1+\frac{1}{x})^x$

1. $\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{3x} = ?$ **à** $\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{3x} = \lim_{x\to+\infty}\left[\left(1+\frac{2}{x}\right)^{x}\right]^{3} = \lim_{x\to+\infty}\left[\left(1+\frac{1}{x/2}\right)^{x}\right]^{3} =$

$\lim_{t\to+\infty}\left[\left(1+\frac{1}{t}\right)^{2t}\right]^{3} = \lim_{t\to+\infty}\left[\left(1+\frac{1}{t}\right)^{t}\right]^{6} = e^{6}$ Fazendo t= $x/2$, $\begin{cases} x\to+\infty \\ t\to+\infty \end{cases}$ e x=2t

2. $\lim_{x\to 0}(1-3x)^{5/x} = ?$ **à** $\lim_{x\to 0}(1-3x)^{5/x} = \lim_{x\to 0}\left[(1-3x)^{1/x}\right]^{5} = \lim_{t\to 0}\left[(1+t)^{-3/t}\right]^{5} =$

$\lim_{t\to 0}\left[(1+t)^{1/t}\right]^{-15} = e^{-15} = \frac{1}{e^{15}}$

Fazendo $t=-3x$, $\begin{cases} x\to 0 \\ t\to 0 \end{cases}$ e $x=-\frac{t}{3}$

3. $\lim_{x\to-\infty}\left(1-\frac{3}{4x}\right)^{1-2x} = ?$ **à** $\lim_{x\to-\infty}\left(1-\frac{3}{4x}\right)^{1-2x} = \lim_{x\to-\infty}\left(1+\frac{1}{-\frac{4x}{3}}\right)^{1-2x} =$

$\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{1+\frac{3t}{2}} = \lim_{x\to-\infty}\left(1+\frac{1}{t}\right)^{1} . \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{\frac{3t}{2}} = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{\frac{3t}{2}}$ $t=-\frac{4x}{3}$, $\begin{cases} x\to-\infty \\ t\to+\infty \end{cases}$ com $x=-\frac{3t}{4}$

4. $\lim_{x\to+\infty}\left(\frac{5x}{2+5x}\right)^{1-2x} = ?$ **à** $\lim_{x\to+\infty}\left(\frac{5x}{2+5x}\right)^{1-2x} = \lim_{x\to+\infty}\left(\frac{5x/5x}{2/5x+5x/5x}\right)^{1-2x} =$

$\lim_{x\to+\infty}\left(\frac{1}{\frac{1}{5x/2}+1}\right)^{1-2x} = \lim_{x\to+\infty}\left(\frac{1}{\frac{1}{5x/2}+1}\right)^{1} . \lim_{x\to+\infty}\left(\frac{1}{\frac{1}{5x/2}+1}\right)^{-2x} =$

$$\lim_{t\to+\infty}\left[\left(\frac{1}{\frac{1}{t}+1}\right)^{\frac{2}{5}.t}\right]^{-2} = \lim_{t\to+\infty}\left[\left(\frac{1}{1+\frac{1}{t}}\right)^{t}\right]^{\frac{2}{5}.(-2)} = \lim_{t\to+\infty}\left[\frac{1}{\left(1+\frac{1}{t}\right)^{t}}\right]^{\frac{-4}{5}.} = \frac{1}{e^{-\frac{4}{5}}} = e^{\frac{4}{5}}$$

Fazendo t= $\frac{5x}{2}$, $\begin{cases} x \to +\infty \\ t \to +\infty \end{cases}$ e x= $\frac{2t}{5}$.

5. $\lim_{x\to+\infty}\left(\frac{x+4}{x-3}\right)^{x}$ =? à $\lim_{x\to+\infty}\left(\frac{x+4}{x-3}\right)^{x} = \lim_{x\to+\infty}\left(\frac{\frac{x+4}{x}}{\frac{x-3}{x}}\right)^{x} = \lim_{x\to+\infty}\left(\frac{\frac{x}{x}+\frac{4}{x}}{\frac{x}{x}-\frac{3}{x}}\right)^{x} =$

$$\lim_{x\to+\infty}\left(\frac{1+\frac{4}{x}}{1-\frac{3}{x}}\right)^{x} = \lim_{x\to+\infty}\left(\frac{1+\frac{1}{x/4}}{1+\frac{1}{x/-3}}\right)^{x} = \frac{e^{4}}{e^{-3}} = e^{7}$$

6. $\lim_{x\to+\infty}\left(\frac{x+a}{x-a}\right)^{x}$ =? à $\lim_{x\to+\infty}\left(\frac{\frac{x+a}{x}}{\frac{x-a}{x}}\right)^{x} = \lim_{x\to+\infty}\left(\frac{\frac{x}{x}+\frac{a}{x}}{\frac{x}{x}-\frac{a}{x}}\right)^{x} = \lim_{x\to+\infty}\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^{x} =$

$$\lim_{x\to+\infty}\left(\frac{1+\frac{1}{x/a}}{1+\frac{1}{x/-a}}\right)^{x} = \frac{\lim_{x\to+\infty}\left(1+\frac{1}{x/a}\right)^{x}}{\lim_{x\to+\infty}\left(1+\frac{1}{x/-a}\right)^{x}} = \frac{e^{a}}{e^{-a}} = e^{a-(-s)} = e^{2a} .$$

7. $\lim_{x\to 0}\sqrt[x]{1+x}$ =? à $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e^{1}$

8. $\lim_{x\to 0}\sqrt[x]{1-3.x}$ =? à $\lim_{x\to 0}\sqrt[x]{1-3.x} = \lim_{t\to+\infty}\left(1+\frac{-3}{t}\right)^{t} = \lim_{t\to+\infty}\left(1+\frac{1}{\frac{t}{-3}}\right)^{t} =$

$$\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^{-3y} = \lim_{y\to-\infty}\left[\left(1+\frac{1}{y}\right)^{y}\right]^{-3} = e^{-3}$$ $\leftarrow$ fazendo $y=-\frac{t}{3}$, $\begin{cases} t \to +\infty \\ y \to -\infty \end{cases}$, com t=-3y

9. $\lim_{x\to 0}\left(1+tg^{2}x\right)^{\cot g^{2}x}$ =? à fazendo $t=\cot g^{2}x$ e substituindo no limite, temos

$$\lim_{x\to 0}\left(1+tg^{2}x\right)^{\cot g^{2}x} = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{t} = e$$

Outros Limites que dependem do limite fundamental $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$

Teorema: Prove que $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$, onde $\ln a$ é o logaritmo de a na base e.

Demonstração: Fazendo $t=a^x-1$, $\begin{cases} x\to 0 \\ t\to 0\end{cases}$

$a^x=t+1$, aplicando-se logaritmo natural na igualdade anterior, temos:

$\ln a^x=\ln(t+1)$ à $x.\ln a=\ln(t+1)$ à

$x=\frac{\ln(t+1)}{\ln a}$ Logo $\lim_{x\to 0}\frac{a^x-1}{x}=$

$$\lim_{t\to 0}\frac{t}{\frac{\ln(t+1)}{\ln a}}=\lim_{t\to 0}\frac{\ln a}{\frac{\ln(t+1)}{t}}=\frac{\lim_{t\to 0}\ln a}{\lim_{t\to 0}\frac{\ln(t+1)}{t}}=$$

$$\frac{\ln a}{\lim_{t\to 0}\ln(t+1)^{1/t}}=\frac{\ln a}{\ln\lim_{t\to 0}(t+1)^{1/t}}=\frac{\ln a}{\ln e}=\ln a.$$

Calcular os limites abaixo: use o limite $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$

10. $\lim_{x\to 0}\frac{a^{3x}-1}{x}=?$ à $\lim_{x\to 0}\frac{a^{3x}-1}{x}=\lim_{t\to 0}\frac{a^t-1}{\frac{t}{3}}=\lim_{t\to 0}3.\frac{a^t-1}{t}=3.\ln a$.

Fazendo $t=3x$, $\begin{cases} x\to 0 \\ t\to 0\end{cases}$ com $x=\frac{t}{3}$.

11. $\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=?$ à $\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=\lim_{x\to 0}\frac{e^x-\frac{1}{e^x}}{2x}=\lim_{x\to 0}\frac{\frac{e^{2x}-1}{e^x}}{2x}=$

$\lim_{x\to 0}\frac{e^{2x}-1}{e^x.2x}=\lim_{x\to 0}\frac{e^{2x}-1}{2x}.\frac{1}{e^x}=\lim_{x\to 0}\frac{e^{2x}-1}{2x}.\lim_{x\to 0}\frac{1}{e^x}=\ln e.\frac{1}{e^0}=1.1=1$

12. $\lim_{x\to 0}\frac{e^x-e^2}{x-2}=?$ à $\lim_{x\to 0}\frac{e^x-e^2}{x-2}=\lim_{x\to 0}\frac{e^2.(e^{x-2}-1)}{x-2}=\lim_{x\to 0}\frac{e^2}{1}.\frac{e^{x-2}-1}{x-2}=$

$e^2.\ln e=e^2$

13. $\lim_{x\to 0}\frac{e^x-e^a}{x-a}=?$ à $\lim_{x\to 0}\frac{e^x-e^a}{x-a}=\lim_{x\to 0}\frac{e^a.(e^{x-a}-1)}{x-a}=$

$\lim_{x\to 0}e^a.\lim_{x\to 0}\frac{e^{x-a}-1}{x-a}=e^a.\ln e=e^a$

14. $\lim_{x\to 0}\frac{e^{\sqrt{x+1}}-e}{x}=?$ à $\lim_{x\to 0}\frac{e^{\sqrt{x+1}}-e}{x}=\lim_{x\to 0}\frac{e.\left(e^{\sqrt{x+1}-1}-1\right)}{x}=$

$\lim_{t\to 0}\frac{e.(e^t-1)}{t^2+2t}=\lim_{t\to 0}\frac{e.(e^t-1)}{t(t+2)}=\lim_{t\to 0}\frac{e}{t+2}.\frac{e^t-1}{t}=\frac{e}{2}.\ln e=\frac{e}{2}$ Fazendo

$t=\sqrt{x+1}-1$ $\begin{cases} x\to 0 \\ t\to 0\end{cases}$, onde $x=t^2+2t$.

15. $\lim_{x\to 0}\frac{5^{2x}-1}{3^x-1}=?$ **à** $\lim_{x\to 0}\frac{5^{2x}-1}{3^x-1}=\lim_{x\to 0}\frac{2.\frac{5^{2x}-1}{2x}}{\frac{3^x-1}{x}}=\frac{\lim_{x\to 0}2.\frac{5^{2x}-1}{2x}}{\lim_{x\to 0}\frac{3^x-1}{x}}=$

$\frac{2}{1}.\frac{\ln 5}{\ln 3}$

16. $\lim_{x\to 4}\frac{3^x-81}{x-4}=?$ **à** $\lim_{x\to 4}\frac{3^x-81}{x-4}=\lim_{x\to 4}\frac{3^x-3^4}{x-4}=\lim_{x\to 4}\frac{3^4.(3^{x-4}-1)}{x-4}=$

$\lim_{x\to 4}3^4.\lim_{x\to 4}\frac{3^{x-4}-1}{x-4}=\ 3^4.\ln 3$

17. $\lim_{x\to 0}\frac{a^x-b^x}{x}=?$ **à** $\lim_{x\to 0}\frac{a^x-b^x}{x}=\lim_{x\to 0}\frac{b^x\left[\left(\frac{a}{b}\right)^x-1\right]}{x}=\lim_{x\to 0}b^x.\frac{\left(\frac{a}{b}\right)^x-1}{x}=$

$\lim_{x\to 0}b^x.\lim_{x\to 0}\frac{\left(\frac{a}{b}\right)^x-1}{x}=b^0.\ln\frac{a}{b}$

18. $\lim_{x\to -2}\frac{2^x-0,25}{2x^2+5x+2}=?$ **à** Solução: $\lim_{x\to -2}\frac{2^x-0,25}{2x^2+5x+2}=$

$\lim_{x\to -2}\frac{2^x-\frac{1}{4}}{2x^2+5x+2}=\lim_{x\to -2}\frac{\frac{1}{4}(2^{x+2}-1)}{(x+2).(2x+1)}=\lim_{x\to -2}\frac{2^{x+2}-1}{x+2}.\frac{1}{4.2.\left(x+\frac{1}{2}\right)}=$

$\lim_{x\to -2}\frac{2^{x+2}-1}{x+2}.\lim_{x\to -2}\frac{1}{4.2.\left(x+\frac{1}{2}\right)}=\ln 2.\frac{1}{4.2.\left(-2+\frac{1}{2}\right)}=\frac{1}{4.2.\left(-\frac{3}{2}\right)}.\ln 2=\frac{1}{4.(-3)}.\ln 2=$

$-\frac{1}{12}.\ln 2$

BriotxRuffini para fatorar: $2x^2+5x+2=?$ $=(x+2).(2x+1)$

	2	5	2
-2	•	-4	-2
	2	1	0 (resto)

19. $\lim_{x\to 0}\frac{e^{x+1}-e}{3x}=?$ **à** $\lim_{x\to 0}\frac{e^{x+1}-e}{3x}=\lim_{x\to 0}\frac{e.(e^x-1)}{3x}=\lim_{x\to 0}\frac{e^x-1}{x}.\frac{e}{3}=$

$\lim_{x\to 0}\frac{e^x-1}{x}.\lim_{x\to 0}\frac{e}{3}=\frac{e}{3}.\ln e=\frac{e}{3}$

20. $\lim_{x\to 0}\frac{x^2-3x}{e^x-1}=?$ à $\lim_{x\to 0}\frac{x^2-3x}{e^x-1}=\lim_{x\to 0}\frac{\frac{x^2-3x}{x}}{\frac{e^x-1}{x}}=\lim_{x\to 0}\frac{x-3}{\frac{e^x-1}{x}}=\frac{\lim_{x\to 0}x-3}{\lim_{x\to 0}\frac{e^x-1}{x}}$

$=\frac{-3}{\ln e}=-3$

21. $\lim_{x\to 0}\frac{\operatorname{sen}3x}{e^x-1}=?$ à $\lim_{x\to 0}\frac{\operatorname{sen}3x}{e^x-1}=\lim_{x\to 0}\frac{3.\frac{\operatorname{sen}3x}{3x}}{\frac{e^x-1}{x}}=\frac{\lim_{x\to 0}3.\frac{\operatorname{sen}3x}{3x}}{\lim_{x\to 0}\frac{e^x-1}{x}}=\frac{3.1}{\ln e}=3$

22. $\lim_{x\to 0}\frac{e^{3x}-1}{\operatorname{sen}4x}=?$ à $\lim_{x\to 0}\frac{e^{3x}-1}{\operatorname{sen}4x}=\lim_{x\to 0}\frac{3.\frac{e^{3x}-1}{3x}}{4.\frac{\operatorname{sen}4x}{4x}}=\frac{\lim_{x\to 0}3.\frac{e^{3x}-1}{3x}}{\lim_{x\to 0}4.\frac{\operatorname{sen}4x}{4x}}=$

$\frac{3.\ln e}{4.1}=\frac{3}{4}$

23. $\lim_{x\to 0}\frac{e^{\operatorname{sen}x}-1}{\cos\left(\frac{p}{2}-x\right)}=?$ à $\lim_{x\to 0}\frac{e^{\operatorname{sen}x}-1}{\cos\left(\frac{p}{2}-x\right)}=\lim_{x\to 0}\frac{e^{\operatorname{sen}x}-1}{\operatorname{sen}x}=\ln e=1$

24. $\lim_{x\to 1}\frac{2^{x+1}-4}{tg(x-1)}=?$ à $\lim_{x\to 1}\frac{2^{x+1}-4}{tg(x-1)}=\lim_{x\to 1}\frac{2^{x+1}-2^2}{tg(x-1)}=\lim_{x\to 1}\frac{2^2.(2^{x-1}-1)}{tg(x-1)}$

$=\lim_{x\to 1}\frac{2^2.\frac{2^{x-1}-1}{x-1}}{\frac{tg(x-1)}{x-1}}=\frac{\lim_{x\to 1}2^2.\frac{2^{x-1}-1}{x-1}}{\lim_{x\to 1}\frac{tg(x-1)}{x-1}}=\frac{\lim_{x\to 1}2^2.\lim_{x\to 1}\frac{2^{x-1}-1}{x-1}}{\lim_{x\to 1}\frac{tg(x-1)}{x-1}}=\frac{2^2.\ln 2}{1}=4.\ln 2$

25. $\lim_{x\to 0}\frac{e^{\operatorname{sen}x-\cos x}-e^{-1}}{1+\operatorname{sen}x-\cos x}=?$ à $\lim_{x\to 0}\frac{e^{\operatorname{sen}x-\cos x}-e^{-1}}{1+\operatorname{sen}x-\cos x}=\lim_{x\to 0}\frac{e^{-1}.(e^{\operatorname{sen}x-\cos x}-1)}{\operatorname{sen}x-\cos x+1}=$

$\lim_{x\to 0}\frac{e^{\operatorname{sen}x-\cos x+1}-1}{e.(\operatorname{sen}x-\cos x+1)}=\lim_{x\to 0}\frac{1}{e}.\frac{e^{\operatorname{sen}x-\cos x+1}-1}{(\operatorname{sen}x-\cos x+1)}=\lim_{t\to 0}\frac{1}{e}.\frac{e^t-1}{t}=\lim_{t\to 0}\frac{e^t-1}{t}.\frac{1}{e}=\frac{1}{e}.\ln e=e^{-1}$

Fazendo $t=1+\operatorname{sen}x-\cos x,\begin{cases}x\to 0\\ t\to 0\end{cases}$

26. $\lim_{x\to 0}\frac{e^x+\operatorname{sen}x-1}{\ln(1+x)}=?$ à $\lim_{x\to 0}\frac{e^x+\operatorname{sen}x-1}{\ln(1+x)}=\lim_{x\to 0}\frac{e^x-1+\operatorname{sen}x}{\ln(1+x)}=$

$\lim_{x\to 0}\frac{\frac{e^x-1}{x}+\frac{\operatorname{sen}x}{x}}{\frac{\ln(1+x)}{x}}=\frac{\lim_{x\to 0}\frac{e^x-1}{x}+\lim_{x\to 0}\frac{\operatorname{sen}x}{x}}{\lim_{x\to 0}\frac{\ln(1+x)}{x}}=\frac{\ln e+1}{\ln\lim_{x\to 0}(1+x)^{\frac{1}{x}}}=\frac{2}{\ln e}=2.$

Aplicando a Regra de L'Hôspital: $\lim_{x\to 0}\frac{e^x+\operatorname{sen}x-1}{\ln(1+x)}=?$ à $\frac{e^0+\operatorname{sen}0-1}{\ln 1}=\frac{0}{0}$

Fazendo $\frac{f(x)}{g(x)} = \frac{e^x + \text{sen}\, x - 1}{\ln(1+x)}$ **à** $\frac{f(0)}{g(0)} = \frac{0}{0}$. Derivando separadamente o numerador e o denominador, temos: $\frac{f'(x)}{g'(x)} = \frac{e^x + \cos x}{\frac{1}{1+x}}$ e $\frac{f'(0)}{g'(0)} = \frac{e^0 + \cos 0}{\frac{1}{1+0}} = \frac{2}{1}$. Logo

$\lim_{x \to 0} \frac{e^x + \text{sen}\, x - 1}{\ln(1+x)} = \frac{2}{1} = 2.$

27. $\lim_{x \to 0} \frac{e^{ax} + \text{sen}\, ax - 1}{\ln(1+ax)} = ?$ **à** $\lim_{x \to 0} \frac{e^{ax} - 1}{1} \cdot \frac{\text{sen}\, ax}{\ln(1+ax)} =$

$$\lim_{x \to 0} \frac{e^x - 1}{x} \cdot \frac{\text{sen}\, ax}{\frac{1}{x} . \ln(1+ax)} = 1 . \lim_{x \to 0} \frac{\text{sen}\, ax}{\frac{1}{x} . \ln(1+ax)} = \lim_{x \to 0} \frac{\text{sen}\, ax}{\frac{1}{x} . \ln(1+ax)} = \lim_{x \to 0} \frac{\text{sen}\, ax}{\ln(1+ax)^{\frac{1}{x}}} =$$

$$\frac{\lim_{x \to 0} \text{sen}\, ax}{\lim_{x \to 0} \ln(1+ax)^{\frac{1}{x}}} = \frac{\lim_{x \to 0} \text{sen}\, ax}{\lim_{x \to 0} \ln(1+ax)^{\frac{1}{x}}} = \frac{\text{sen}(a.0)}{\ln \lim_{x \to 0} (1+ax)^{\frac{1}{x}}} = \frac{0}{e^a} = 0$$