A geometric proof of Cavalieri's quadrature formula

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Norman Wildberger [1] recently gave a completely algebraic argument, i.e., not using antidifferentiation or Riemann sums, of Cavaleri's formula

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1}. \tag{1}$$

The following is a completely geometric proof of this formula. The basic observation is that formula (1) is essentially the general statement

(2) *The volume of an n–dimensional cone = base × height/n.*

Here "n–dimensional cone" means the set of line segments connecting the base of the cone, which lies on an $n-1$–dimensional plane, to its vertex, which is a point not lying on this plane. In fact, formula (1) follows from any *single* case of the cone result, so one only needs to find the simplest example. The following appears to satisfy this, where I prove (1) with $n+1 \Rightarrow n$ in order to highlight the volume argument.

Consider an n–dimensional cube of side $2a$ and let C be the cone (pyramid) with vertex at the center of the cube and base an $n-1$–dimensional face (cell) of the cube. Since the cube has $2n$ such faces, the volume of C is $1/(2n)$ the volume of the cube. Since, by definition, the volume of the cube is $(2a)^n$, one gets

$$\text{volume of } C = \frac{2^{n-1}a^n}{n}. \tag{3}$$

On the other hand, note that C is a pyramid whose base has $n-1$–dimensional volume $(2a)^{n-1}$ and is at height a from the vertex. By similarity, it follows that the section of C which is at height x from the vertex will have volume $(2x)^{n-1}$. The volume of C is the sum of all such sections, so that, by Cavalieri's infinitesimal definition of integral, one has

$$\text{volume of } C = \int_0^a (2x)^{n-1} dx. \tag{4}$$

Formula (1) follows upon comparing (3) and (4).

Finally, note that one can prove that C satisfies the cone formula (2) without using any algebraic manipulation: The volume of the cube is the product of all the sides meeting at a vertex, so its volume equals base×height. Since C has the same base as the cube, half the height, and $1/(2n)$ the volume, one concludes that the volume of C is base×height/n.

References

[1] N. Wildberger, *A new proof of Cavalieri's Quadrature Formula,* American Math. Monthly **109**, November 2002.

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