A shorthand notation—Augmented Matrix for the coefficients of a System of Linear Equations

For example:

$$\begin{array}{rrrrrrrr} 3x & + & 2y & - & z & = & -16 \\ 6x & - & 4y & + & 3z & = & 12 \\ 5x & - & 2y & + & 2z & = & 4 \end{array} \qquad \left[\begin{array}{rrr|r} 3 & 2 & -1 & -16 \\ 6 & -4 & 3 & 12 \\ 5 & -2 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r|rrrrrrr} & 3x & + & 2y & - & z & = & -16 \\ R_2 - 2R_1 & & - & 8y & + & 5z & = & 44 \\ 3R_3 - 5R_1 & & - & 16y & + & 11z & = & 92 \end{array} \qquad \begin{array}{r} \\ R_2 - 2R_1 \\ 3R_3 - 5R_1 \end{array} \left[\begin{array}{rrr|r} 3 & 2 & -1 & -16 \\ 0 & -8 & 5 & 44 \\ 0 & -16 & 11 & 92 \end{array}\right]$$

$$\begin{array}{r|rrrrrrr} & 3x & + & 2y & - & z & = & -16 \\ & & - & 8y & + & 5z & = & 44 \\ R_3 - 2R_2 & & & & + & z & = & 4 \end{array} \qquad \begin{array}{r} \\ \\ R_3 - 2R_2 \end{array} \left[\begin{array}{rrr|r} 3 & 2 & -1 & -16 \\ 0 & -8 & 5 & 44 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Gauss-Jordan Method

Use row operations to bring the augmented matrix into a "**row reduced** **echelon form**"

$$\left[\begin{array}{cccc|c} 1 & 2 & 5 & 10 & 30 \\ 0 & 1 & 2 & 2 & 16 \\ 0 & 0 & 1 & -2 & 7 \end{array}\right]$$

is only **echelon**.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 8 & -9 \\ 0 & 1 & 0 & 6 & 2 \\ 0 & 0 & 1 & -2 & 7 \end{array}\right]$$

is more than echelon! It is "**row-reduced** **echelon**".

- It is echelon.
- Moreover, the leading non-zero coefficient of each row is the only non-zero coefficient in that column.

Re-introducing the variables, this is the same as

$$\begin{cases} x \qquad\qquad + 8w = -9 \\ \qquad y \qquad + 6w = 2 \\ \qquad\qquad z - 2w = 7 \end{cases} \quad \text{i.e.} \quad \begin{cases} x = -9 - 8w \\ y = 2 - 6w \\ z = 7 + 2w \end{cases}$$

Thus the solutions are $(-9-8w,\ \ 2-6w,\ \ 7+2w,\ \ w)$, where w can take on any value. So solutions can be read off *directly* from the row-reduced echelon form without back-substitution!

Example 1: (2.2, #34)

$$\begin{cases} 3x + 2y - z = -16 \\ 6x - 4y + 3z = 12 \\ 5x - 2y + 2z = 4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -16 \\ 6 & -4 & 3 & 12 \\ 5 & -2 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ R_2 - 2R_1 \\ 3R_3 - 5R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -16 \\ 0 & -8 & 5 & 44 \\ 0 & -16 & 11 & 92 \end{array}\right]$$

$$\begin{array}{r} 4R_1 + R_2 \\ \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 12 & 0 & 1 & -20 \\ 0 & -8 & 5 & 44 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{r} R_1 - R_3 \\ R_2 - 5R_3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 12 & 0 & 0 & -24 \\ 0 & -8 & 0 & 24 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{r} R_1 / 12 \\ -R_2 / 8 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Answer: $x = -2$, $y = -3$, $z = 4$

Example 2:

$$\begin{cases} & & z + u + 2v = 1 \\ x + 2y + z - v = -2 \\ 3x + 6y + 4z - 2v = -7 \\ 2x + 4y + z - 3v = -3 \\ x + 2y + 3z + u + 2v = -2 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & -1 & -2 \\ 3 & 6 & 4 & 0 & -2 & -7 \\ 2 & 4 & 1 & 0 & -3 & -3 \\ 1 & 2 & 3 & 1 & 2 & -2 \end{array}\right]$$

$$\begin{array}{c} R_2 \\ R_1 \\ \\ \\ \\ \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 3 & 6 & 4 & 0 & -2 & -7 \\ 2 & 4 & 1 & 0 & -3 & -3 \\ 1 & 2 & 3 & 1 & 2 & -2 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3 - 3R_1 \\ R_4 - 2R_1 \\ R_5 - R_1 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 & 3 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \\ R_3 - R_2 \\ R_4 + R_2 \\ R_5 - 2R_2 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & -3 & -3 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_3 \\ R_2 + R_3 \\ \\ R_4 + R_3 \\ R_5 - R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ \\ -R_3 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & u & v \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

So

$$\begin{cases} x = -1-2y+2v \\ z = -1-v \\ u = 2-v \end{cases}$$ with y and v free to take on any value.

Example 3: (The Example 7 in our discussion on echelon method.)

$$\begin{cases} x + y + z = 40 \\ 2x + 3y - z = 10 \end{cases}$$

Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 40 \\ 2 & 3 & -1 & 10 \end{array}\right]$$

$$\begin{array}{r} \\ R_2 - 2R_1 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 40 \\ 0 & 1 & -3 & -70 \end{array}\right]$$

$$\begin{array}{r} R_1 - R_2 \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 4 & 110 \\ 0 & 1 & -3 & -70 \end{array}\right] \end{array}$$

So

$x = 110 - 4z$

$y = -70 + 3z$

with z free to take on any value.

Answer: Infinitely many solutions $(110-4z, -70+3z, z)$, where z can take on any value.

Example 3: (Example 9 of our discussion of the echelon method.)

$$\begin{cases} x + y + z = 4 \\ 2x + 4y - z = 4 \\ 4x + 6y + z = 17 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 4 & -1 & 4 \\ 4 & 6 & 1 & 17 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & -3 & -4 \\ 0 & 2 & -3 & 1 \end{array}\right]$$

$$\begin{array}{c} 2R_1 - R_2 \\ \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} x & y & z & \\ 2 & 0 & 5 & 12 \\ 0 & 2 & -3 & -4 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

The 3rd equation here reads $0 = 5$.
Contradiction.
Answer: Now solution.

Application Problems

Example [2.2, #46] (Manufacturing, simple)
Fred's Furniture Factory has 1950 machine hours available each week in the cutting department, 1490 hours in the assembly department, and 2160 in the finishing department. Manufacturing a chair requires 0.2 hours of cutting, 0.3 hours of assembly, and 0.1 hours of finishing. A cabinet requires 0.5 hours of cutting, 0.4 hours of assembly, and 0.6 hours of finishing. A buffet requires 0.3 hours of cutting, 0.1 hours of assembly, and 0.4 hours of finishing. How many chairs, cabinets, and buffets should be produced in order to use all the available production capacity?

Answer: (2000, 1600, 2500)

Example [2.2, #51] (two plants, two dealerships)
An auto manufacturer sends cars from two plants, I and II, to dealerships A and B located in a Midwestern city. Plant I has a total of 28 cars to send, and plant II has 8. Dealer A needs 20 cars, and dealer B nees 16. Transportation costs per car, based on the distance of each dealership from each plant, are $220 from I to A, $300 from I to B, $400 from II to A, $180 from II to B. The manufacturer wants to limit transportation costs to $10,640. How many cars should be sent from each plant to each of the two dealerships?

Answer: I-A 12, I-B 16, II-A 8, II-B 0

Example [2.2, #47] (Manufacturing, three subproblems)
Nadir Inc. produces three models of television sets: deluxe, super-deluxe, and ultra. Each deluxe set requires 2 hours of electroics work, 2 hours of assembly time, and 1 hour of finishing time. Each super-deluxe requires 1, 3, and 1 hour of electronics, assembly, and finishing time, respectively. Each ultra requires 3, 2, and 2 hours of the same work, respectively.

a. There are 100 hours available for electronics, 100 hours available for assembly, and 65 hours available for finishing per week. How many of each model should be produced each week if all available time is to be used?
b. Suppose everything is the same as in part a, but an ultra set requires 6, rather than 3, hours of electronics work. How many solutions are there now?
c. Suppose everything is the same as in part b, but the total hours available for electronics changes from 100 hours to to 160 hours. How many solutions are there?

   Answer: (a) (15, 10, 20) (b) No solution. (c) $d = 95 - 4u$, $s = 2u - 30$, with $15 \le u \le 23$, so 9 solutions.

Example [2.2, #56] (Traffic Control)

Answer: (b) $x_1 = 1000 - x_4$, $x_2 = 100 + x_4$, $x_3 = 600 - x_4$ (c) $0 \le x_4 \le 600$

(d) $400 \le x_1 \le 1000$, $100 \le x_2 \le 700$, $0 \le x_3 \le 600$