Chapter 2: Systems of Linear Equations and Matrices

Chapter Highlight:

- Echelon Method   (2.1)
- Gauss-Jordan Method (2.2)
- Addition and Subtraction of Matrices (2.3)
- Multiplication of Matrices (2.4)
- Matrix Inverses (2.5)
- Input-Output Model (2.6)

You should be well versed in Gauss-Jordan Method in order to fare well when we get to linear programming.

Systems of Linear Equations (a.k.a. **Systems of First-Degree Equations**)

Echelon Method

Example 1:

$$\begin{cases} x+2y=5 \\ 2x+y=-2 \end{cases}$$

Example 2:

$$\begin{cases} x+2y=5 \\ 3x+6y=4 \end{cases}$$

Example 3:

$$\begin{cases} x+2y=5 \\ 3x+6y=15 \end{cases}$$

Example 4:

$$\begin{cases} x + 0.5y - 0.5z = 1 \\ \quad\quad y + z = 0 \\ \quad\quad\quad\quad z = 1 \end{cases}$$

Example 5:

$$\begin{cases} 2x + y - z = 2 \\ x + 3y + 2z = 1 \\ 2x + 4y + 3z = 3 \end{cases}$$

Example 6:

$$\begin{cases} x + y + z = 40 \\ \quad\quad y - 3z = -70 \end{cases}$$

These two systems are of "**echelon form**": i.e. each successive row has more leading empty slots on the left of the equals sign than the row before it does.

A system in echelon form is easily dealt with by "back-substitution": start with the very last equation and work our way upwards.

The idea of echelon method is to make every effort to bring a system into an equivalent one in echelon form.

Example 7:

$$\begin{cases} x + y + z = 40 \\ 2x + 3y - z = 10 \end{cases}$$

Example 8:

$$\begin{cases} x + y + z = 4 \\ 2x + 4y - z = 4 \\ 4x + 6y + z = 12 \end{cases}$$

Example 9:

$$\begin{cases} x + y + z = 4 \\ 2x + 4y - z = 4 \\ 4x + 6y + z = 17 \end{cases}$$

Example 10:

$$\begin{cases} 3x + 2y - z = -16 \\ 6x - 4y + 3z = 12 \\ 5x - 2y + 2z = 4 \end{cases}$$

Transformations of a Linear System ("Row Operations"):

1. **Exchanging any two equations.**
2. **Multiplying both sides of an equation by a nonzero number.**
3. **Replacing any equation by a nonzero multiple of that equation plus a multiple of another equation.**

Note that all these operations are ***reversible***!!!

Subjecting a system to a sequence of row operations results in a system that is *equivalent* to the original system.

Echelon Method: use row operations systematically to bring the system to an equivalent system that is in echelon form.

Example 1:

$$\begin{cases} x+2y=5 \\ 2x+y=-2 \end{cases}$$

Let's solve this system as in Math 120 or Algebra II
One approach is the substitution method.
The 1st equation is the same as saying that $x=5-2y$.
This helps us rewrite the 2nd equation:
$2[5-2y]+y=-2$, that is, $10-3y=-2$, which is the same as
$12=3y$. This means $y=4$
Then we go back to use $x=5-2y$ to find x: $x=5-2(4)=-3$
Answer: $x=-3$, $y=4$.

Let's render this substitution strategy into the following language:

	x	$+$	$2y$	$=$	5
	$2x$	$+$	y	$=$	-2

R_1	x	$+$	$2y$	$=$	5
R_2-2R_1		$-$	$3y$	$=$	-12

Adding to R_2 just the right multiple of R_1 to let x disappear is the same as taking the expression of x that R_1 offers and substitute it into R_2.

At this stage, the system has been brought to an "**echelon form**":
therefore it can be solved by back-substitution:
Start with the very last equation and work our way back up.
The last equation $-3y=-12$ is the same as saying $y=4$.
Go back one line, we replace y by 4 in $x+2y=5$ to get $x+2(4)=5$, which is the same as saying $x=-3$. Thus we are done!!!

Example 2:

$$\begin{cases} x+2y=5 \\ 3x+6y=4 \end{cases}$$

Follow the same procedure, we would get

	x	$+$	$2y$	$=$	5
	$3x$	$+$	$6y$	$=$	4

R_1	x	$+$	$2y$	$=$	5
R_2-3R_1			0	$=$	-11

But this time we see that the second equation is clearly absurd. There is therefore no solutions!!

Example 3:

$$\begin{cases} x + 2y = 5 \\ 3x + 6y = 15 \end{cases}$$

Follow the same procedure, we would get

$$\begin{array}{c|ccccc} & x & + & 2y & = & 5 \\ & 3x & + & 6y & = & 15 \end{array}$$

$$\begin{array}{c|ccccc} R_1 & x & + & 2y & = & 5 \\ R_2 - 3R_1 & & & 0 & = & 0 \end{array}$$

But this time we see that the second equation is trivially true—it doesn't add any impose any constraint! So, effectively, the system only requires $x + 2y = 5$. This is the same as saying $x = 5 - 2y$.

Answer: There are infinitely many solutions: $(5 - 2y, \quad y)$, where y can take on any value.

Example 4:

$$\begin{cases} x + 0.5y - 0.5z = 1 \\ y + z = 0 \\ z = 1 \end{cases}$$

This is already in echelon form, so we can start doing back-substitution:
The 3rd equation gives $z = 1$ for free.
The 2nd equation becomes $y + 1 = 0$, therefore $y = -1$.
The 1st equation becomes $2x + (-1) - 1 = 2$, therefore $x = 2$.

Answer: $x = 2$, $y = -1$, $z = 1$

Example 5:

$$\begin{cases} 2x + y - z = 2 \\ x + 3y + 2z = 1 \\ 2x + 4y + 3z = 3 \end{cases}$$

$$\begin{array}{r|l} & 2x + y - z = 2 \\ R_2 - 0.5R_1 & 2.5y + 2.5z = 0 \\ R_3 - R_1 & 3y + 4z = 1 \end{array}$$

$$\downarrow$$

$$\begin{array}{r|l} & 2x + y - z = 2 \\ (1/2.5)R_2 & y + z = 0 \\ & 3y + 4z = 1 \end{array}$$

$$\downarrow$$

$$\begin{array}{r|l} & 2x + y - z = 2 \\ & y + z = 0 \\ R_3 - 3R_2 & z = 1 \end{array}$$

If you want to avoid fractions:

$$\begin{array}{r|l} & 2x + y - z = 2 \\ 2R_2 - R_1 & 5y + 5z = 0 \\ R_3 - R_1 & 3y + 4z = 1 \end{array}$$

$$\downarrow$$

$$\begin{array}{r|l} & 2x + y - z = 2 \\ (1/5)R_2 & y + z = 0 \\ & 3y + 4z = 1 \end{array}$$

$$\downarrow$$

$$\begin{array}{r|l} & 2x + y - z = 2 \\ & y + z = 0 \\ R_3 - 3R_2 & z = 1 \end{array}$$

We have achieved an echelon form, so we can start doing back-substitution:
The 3rd equation gives $z = 1$ for free.
The 2nd equation becomes $y + 1 = 0$, therefore $y = -1$.
The 1st equation becomes $2x + (-1) - 1 = 2$, therefore $x = 2$.

Answer: $x = 2$, $y = -1$, $z = 1$

Example 6:

$$\begin{cases} x + y + z = 40 \\ \quad\;\; y - 3z = -70 \end{cases}$$

This is already in **echelon form**. So we simply do back-substitution.
Start with the 2nd equation, we use it to express the variable y in the first non-vanishing term in terms of the variables that follow it. I.e. $y = -70 + 3z$.
Go back one line, Substitute $-70 + 3z$ for y in the $x + y + z = 40$ to get $x + (-70 + 3z) + z = 40$, which says $x = 110 - 4z$. We are done:

Answer: Infinitely many solutions $(110 - 4z, -70 + 3z, z)$, where z can take on any value.

Example 7:

$$\begin{cases} x + y + z = 40 \\ 2x + 3y - z = 10 \end{cases}$$

$$\begin{array}{c|ccccccc} & x & + & y & + & z & = & 40 \\ & 2x & + & 3y & - & z & = & 10 \end{array}$$

$$\begin{array}{c|ccccccc} & x & + & y & + & z & = & 40 \\ R_2 - 2R_1 & & & y & - & 3z & = & -70 \end{array}$$

This is in echelon form. So we do back-substitution.

Answer: Infinitely many solutions $(110 - 4z, -70 + 3z, z)$, where z can take on any value.

Example 8:

$$\begin{cases} x + y + z = 4 \\ 2x + 4y - z = 4 \\ 4x + 6y + z = 12 \end{cases}$$

$$\left|\begin{array}{l} x + y + z = 4 \\ 2x + 4y - z = 4 \\ 4x + 6y + z = 12 \end{array}\right.$$

$$\begin{array}{l|l} & x + y + z = 4 \\ R_2 - 2R_1 & 2y - 3z = -4 \\ R_3 - 4R_1 & 2y - 3z = -4 \end{array}$$

$$\begin{array}{l|l} & x + y + z = 4 \\ & 2y - 3z = -4 \\ R_3 - R_2 & 0 = 0 \end{array}$$

This is in echelon form.
So the 3rd equation says something trivially true. Therefore we essentially has only two conditions imposed.
The 2nd equation gives $y = -2 + 1.5z$.
Substitute this into the 1st equation gives $x + [-2 + 1.5z] + z = 4$, which says $x = 6 - 2.5z$
Answer: $(6 - 2.5z, \quad -2 + 1.5z, \quad z)$, where z can take on any value.

Example 9:

$$\begin{cases} x + y + z = 4 \\ 2x + 4y - z = 4 \\ 4x + 6y + z = 17 \end{cases}$$

$$\left|\begin{array}{l} x + y + z = 4 \\ 2x + 4y - z = 4 \\ 4x + 6y + z = 17 \end{array}\right.$$

$$\begin{array}{l|l} & x + y + z = 4 \\ R_2 - 2R_1 & 2y - 3z = -4 \\ R_3 - 4R_1 & 2y - 3z = 1 \end{array}$$

$$\begin{array}{l|l} & x + y + z = 4 \\ & 2y - 3z = -4 \\ R_3 - R_2 & 0 = 5 \end{array}$$

The 3rd equation $0 = 5$ is absurd. No solutions.

Example 10:

$$\begin{cases} 3x + 2y - z = -16 \\ 6x - 4y + 3z = 12 \\ 5x - 2y + 2z = 4 \end{cases}$$

Actually two steps!

$$\left.\begin{array}{rrrrrrr} 3x & + & 2y & - & z & = & -16 \\ 6x & - & 4y & + & 3z & = & 12 \\ 5x & - & 2y & + & 2z & = & 4 \end{array}\right.$$

$$\left.\begin{array}{rrrrrrr} 3x & + & 2y & - & z & = & -16 \\ 6x & - & 4y & + & 3z & = & 12 \\ 5x & - & 2y & + & 2z & = & 4 \end{array}\right.$$

$$\begin{array}{r|rrrrrrr} & 3x & + & 2y & - & z & = & -16 \\ R_2 - 2R_1 & & - & 8y & + & 5z & = & 44 \\ & 5x & - & 2y & + & 2z & = & 4 \end{array}$$

$$\begin{array}{r|rrrrrrr} & 3x & + & 2y & - & z & = & -16 \\ R_2 - 2R_1 & & - & 8y & + & 5z & = & 44 \\ 3R_3 - 5R_1 & & - & 16y & + & 11z & = & 92 \end{array}$$

$$\begin{array}{r|rrrrrrr} & 3x & + & 2y & - & z & = & -16 \\ & & - & 8y & + & 5z & = & 44 \\ 3R_3 - 5R_1 & & & -16y & + & 11z & = & 92 \end{array}$$

$$\begin{array}{r|rrrrrrr} & 3x & + & 2y & - & z & = & -16 \\ & & - & 8y & + & 5z & = & 44 \\ R_3 - 2R_2 & & & & + & z & = & 4 \end{array}$$

$$\begin{array}{r|rrrrrrr} & 3x & + & 2y & - & z & = & -16 \\ & & - & 8y & + & 5z & = & 44 \\ R_3 - 2R_2 & & & & + & z & = & 4 \end{array}$$

So the 3rd equation gives $z = 4$.
The 2nd equation then reads $-8y + 5(4) = 44$, so $y = -3$.
Substitute $z = 4$ and $y = -3$ into the 1st equation, we get $3x + 2(-3) - 4 = -16$, so $x = -2$.
Answer: $x = -2$, $y = -3$, $z = 4$.

Note that R_1 participated in changing R_2, but R_1 itself is unchanged!!

Problems for Practice in Class:

Problem 1: (2.1, #24)

$$\begin{cases} 2x & + & y & + & z & = & 9 \\ -x & - & y & + & z & = & 1 \\ 3x & - & y & + & z & = & 9 \end{cases}$$

Answer: $(2, \quad 1, \quad 4)$