Group Classification of Linear Second-Order Delay Ordinary Differential Equation

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Abstract. The linear delay ordinary differential equation

$$y''(x) + a(x)y'(x) + b(x)y'(x-\tau) + c(x)y(x) + d(x)y(x-\tau) = g(x)$$

is studied, where the coefficients $a(x), b(x), c(x)$ and $d(x)$ and function $g(x)$ are arbitrary. In this manuscript, group analysis is applied to find equivalent symmetries of the equation.

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1. Introduction

Let us consider a linear second-order delay ordinary differential equation

$$y''(x) + a(x)y'(x) + b(x)y'(x-\tau) + c(x)y(x) + d(x)y(x-\tau) = g(x). \tag{1.1}$$

For brevity, the symbol y_τ will be used to denote $y(x-\tau)$, y to denote $y(x)$ and y', y'_τ will mean the first derivatives of y at point x and $x-\tau$, respectively. Then equation (1.1) can be simply written as

$$y'' + a(x)y' + b(x)y'_\tau + c(x)y + d(x)y_\tau = g(x). \tag{1.2}$$

Here it is assumed that $b^2 + c^2 \neq 0$.

Application of this equation can be found in biology, physics and engineering, where it is used to model natural phenomena.

2. Lie Group of Transformations

In [**6**], a definition of admitted Lie group of transformations for delay differential equations was developed. A theory of equivalence Lie groups can be considered similar to that of admitted Lie groups. Here this analysis is developed.

Let $\varphi : \Omega \times \triangle \to \Omega$ be a transformation, where Ω is a set of variables $(x, y, \phi), \phi = (a, b, c, d, g)$ and $\triangle \subset \mathbb{R}$ is a symmetric interval with respect to zero. The variable $\epsilon \in \triangle$ is considered as a parameter of the transformation φ. This transform maps the variables (x, y, ϕ) to variables $(\bar{x}, \bar{y}, \bar{\phi})$. Let $\varphi(x, y, \phi; \epsilon)$ be denoted by $\varphi_\epsilon(x, y, \phi)$. The set of functions φ_ϵ forms *a one-parameter transformation Lie group* of the space Ω if it contains the identity transformation as well as inverse of its elements and their composition [**4, 5, 6**]. Alternatively, the notation $\bar{x} = \varphi^x(x, y, \phi; \epsilon), \bar{y} = \varphi^y(x, y, \phi; \epsilon), \bar{\phi} = \varphi^\phi(x, y, \phi; \epsilon)$ is used instead of $\varphi_\epsilon = (\bar{x}, \bar{y}, \bar{\phi})$. The transformed variable y with delay term, it's derivatives and it's derivatives with delay term are defined by $\bar{y}_\tau = \bar{y}(\bar{x} - \tau)$, $\bar{y}' = \dfrac{d\bar{y}}{d\bar{x}}$ and $\bar{y}'_\tau = \dfrac{d\bar{y}}{d\bar{x}}(\bar{x} - \tau)$, respectively.

Let us consider a delay differential equation

$$F(x, y, y_\tau, y', y'_\tau, y'', \phi) = 0. \tag{2.1}$$

A Lie group of transformations is called *admitted* if each transformation maps a solution of the differential equation to a solution of the same equation. Such transformation are called *symmetries*. For this reason, equations for defining symmetries were constructed under the assumption that a Lie group of transformations maps a solution of a delay differential equation into a solution of the same equation. This assumption leads to

$$\frac{\partial F(\bar{x}, \bar{y}, \bar{y}_\tau, \bar{y}', \bar{y}'_\tau, \bar{y}'', \bar{\phi})}{\partial \epsilon}\Big|_{\epsilon=0} = \tilde{X} F(x, y, y_\tau, y', y'_\tau, y'', \phi)\Big|_{(1.2)} = 0. \tag{2.2}$$

Here, the operator $\tilde{X}$ is defined by

$$\tilde{X} = (\eta - y'\xi)\partial_y + (\eta_\tau - y'_\tau\xi_\tau)\partial_{y_\tau} + \eta^{y'}\partial_{y'} + \eta^{y'_\tau}\partial_{y'_\tau} + \eta^{y''}\partial_{y''} + (\zeta - \phi_x\xi)\partial_\phi + \zeta^{\phi_y}\partial_{\phi_y}.$$

where

$$\xi(x, y, \phi) = \frac{\partial \varphi^x}{\partial \epsilon}(x, y, \phi; 0), \qquad \eta(x, y, \phi) = \frac{\partial \varphi^y}{\partial \epsilon}(x, y, \phi; 0),$$

$$\zeta(x, y, \phi) = \frac{\partial \varphi^\phi}{\partial \epsilon}(x, y, \phi; 0), \qquad \xi_\tau(x, y, \phi) = \xi(x - \tau, y_\tau, \phi_\tau),$$

$$\eta^{y'} = D_x(\eta - y'\xi), \qquad \eta^{y''} = D_x(\eta^{y'}),$$

$$\eta_\tau(x, y, \phi) = \eta(x - \tau, y_\tau, \phi_\tau), \;\; \eta^{y'_\tau} = D_x(\eta_\tau - y'_\tau\xi_\tau), \;\; \zeta^{\phi_y} = D_y(\zeta - \phi_x\xi),$$

$$\begin{aligned} D_x &= \partial_x + y'\partial_y + y''\partial_{y'} + \cdots + y'_\tau\partial_{y_\tau} \\ &\quad + (\phi_x + \phi_y y')\partial_\phi + (\phi_{xx} + \phi_{xy} y')\partial_{\phi_x}, \\ D_y &= \partial_y + \phi_y\partial_\phi + \phi_{xy}\partial_{\phi_x} + \ldots \end{aligned}$$

The operator $\tilde{X}$ is called *a canonical Lie-Bäcklund infinitesimal operator* of a symmetry. Lie's theory [**4, 5, 6**] shows that there is a one-to-one correspondence between generators and symmetries. This operator is also equivalent to *an infinitesimal generator* [**6**]

$$X = \xi\partial_x + \eta\partial_y + \zeta\partial_\phi.$$

Equation (2.2) gives a definition of an action of the infinitesimal generator of a Lie group onto a delay differential equation.

2.1. Determining Equations.

Definition 2.1. A Lie group of transformations is called *admitted* if it satisfies the equation

$$\tilde{X} F(x, y, y_\tau, y', y'_\tau, y'', \phi)\Big|_{(2.1)} = 0. \tag{2.3}$$

for any solution of (2.1).

Equation (2.3) is called *a determining equation*. For solving the determining equation one can use the theory of existence of a solution of an initial value problem for delay equation (1.2) [**1**]. This problem is formulated as follows. Let a function $\chi(x), x \in (x_0 - \tau, x_0)$ be given. Find a solution $y(x), x \in [x_0, x_0 + \tilde{\epsilon})$ which satisfies the condition

$$y(x) = \chi(x), \;\; x \in (x_0 - \tau, x_0).$$

Because the initial values are arbitrary, the variables y, y_τ and their derivatives can be considered as arbitrary elements. Thus, if the determining equation (2.3) is written as a polynomial of variables and their derivatives, the coefficients of these variables in the equations must vanish. The method for obtaining the overdetermined system of equations is called *splitting the determining equation.* This gives an overdetermined system of partial differential equations for the coefficients of the infinitesimal generator. The unknown functions ξ, η and ζ can be found by solving this system.

2.2. Equivalence Problem. The problem of finding all equations, which are equivalent to a given equation is called *an equivalence problem.* If the given equation is a *linear* equation, then the equivalence problem is called *a linearization problem.* In this section the importance of an equivalence Lie group of transformations is shown.

Consider a linear second order ordinary differential equation

$$y'' + a(x)y' + c(x)y = g(x). \tag{2.4}$$

A Lie group of transformations of the independent variables, dependent variables and coefficients, which conserves the differential structure of the equation is called *an equivalent Lie group.* This group allows simplifying the coefficients of equations.

For example, S. Lie showed that any linear second order ordinary differential equation (2.4) is equivalent to the equation

$$y'' = 0. \tag{2.5}$$

Equation (2.5) admits the eight-dimensional Lie algebra spanned by the generators

$$X_1 = \frac{\partial}{\partial_x}, X_2 = \frac{\partial}{\partial_y}, X_3 = x\frac{\partial}{\partial_x}, X_4 = y\frac{\partial}{\partial_x}, X_5 = x\frac{\partial}{\partial_y},$$
$$X_6 = y\frac{\partial}{\partial_y}, X_7 = x^2\frac{\partial}{\partial_x} + xy\frac{\partial}{\partial_y}, X_8 = xy\frac{\partial}{\partial_x} + y^2\frac{\partial}{\partial_y}.$$

If one tries to find an admitted Lie group for equation (2.4), then the system of determining equations consists of four second-order ordinary differential equations. In general, this system cannot be solved. The purpose of this manuscript is to do group classification of equation (1.2).

3. Equivalence Symmetries of (1.2)

Let y_p be a particular solution of equation (1.2). Considering the change $\tilde{x} = x, \tilde{y} = y - y_p$, equation (1.2) is reduced to the equation

$$y'' + a(x)y' + b(x)y'_\tau + c(x)y + d(x)y_\tau = 0.$$

Similar to a second-order ordinary differential equation the coefficient $a(x)$ can be reduced by change $y = v(x)q(x)$ with q satisfying the equation $2q' + aq = 0$. We will consider equivalence symmetries of equation

$$y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau = 0. \tag{3.1}$$

instead of (1.2). Letting $F = y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau$, then equation (2.3) becomes

$$\tilde{X}\left(y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau\right)\Big|_{(3.1)} = 0. \tag{3.2}$$

Splitting this equation with respect to $y', y'_\tau, y'_{2\tau}, b'', c'', d'', b', c', d'$ and later with respect to y, y_τ, one finds

$$\xi = \xi_\tau = \alpha, \eta = \beta y + \gamma, \zeta^b = b(-\alpha' + \beta - \beta_\tau),$$
$$\zeta^c = -\beta'' - 2c\alpha', \zeta^d = -b\beta'_\tau - 2d\alpha' + d\alpha - d\alpha_\tau,$$

where $\alpha(x), \beta(x)$ are arbitrary periodic functions with period τ, and $\gamma(x)$ is an arbitrary solution of (3.1). Thus the equivalence symmetry is

$$\bar{x} = \alpha(x), \ \bar{y} = \beta(x)y + \gamma(x). \tag{3.3}$$

4. Group Classification of Equation (3.1)

A Lie group of transformations admitted by equation (3.1) has to satisfy the following determining equation [**3**]

$$\tilde{Y}(y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau)\Big|_{(3.1)} = 0. \tag{4.1}$$

Here, the operator $\tilde{Y}$ is defined by

$$\tilde{Y} = (\eta - y'\xi)\partial_y + (\eta_\tau - y'_\tau\xi_\tau)\partial_{y_\tau} + \eta^{y'}\partial_{y'} + \eta^{y'_\tau}\partial_{y'_\tau} + \eta^{y''}\partial_{y''}.$$

where

$$\xi(x,y) = \frac{\partial\varphi^x}{\partial\epsilon}(x,y;0), \quad \eta(x,y) = \frac{\partial\varphi^y}{\partial\epsilon}(x,y;0),$$
$$\xi_\tau(x,y) = \xi(x-\tau, y_\tau), \quad \eta_\tau(x,y) = \eta(x-\tau, y_\tau),$$
$$\eta^{y'} = D_x(\eta - y'\xi), \quad \eta^{y'_\tau} = D_x(\eta_\tau - y'_\tau\xi_\tau), \ \eta^{y''} = D_x(\eta^{y'}),$$
$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots + y'_\tau\partial_{y_\tau} + y''_\tau\partial_{y'_\tau}.$$

Splitting this equation with respect to $y', y'_\tau, y'_{2\tau}$ and later with respect to y, y_τ, one finds

$$\xi = \xi_\tau, \ \eta = \beta y + \gamma, \tag{4.2}$$
$$\xi_{xx} = 2\beta', \ \beta'' = -c'\xi - 2c\xi_x, \tag{4.3}$$
$$\gamma'' = -b\gamma'_\tau - c\gamma - d\gamma_\tau, \tag{4.4}$$
$$b(\beta - \beta_\tau) = b'\xi + \xi_x b, \tag{4.5}$$
$$d(\beta - \beta_\tau) = d'\xi + b\beta'_\tau + 2\xi_x d. \tag{4.6}$$

By integrating (4.3), one finds $\beta = \xi_x/2 + C_1$, where C_1 is an arbitrary constant. Since $\xi = \xi_\tau$, it implies $\beta = \beta_\tau$. Hence, integrating equation (4.5) one has

$$b\xi = C_2, \tag{4.7}$$

C_2 is an arbitrary constant. Equation (4.6) is written as

$$d'\xi + 2\xi_x d = -\frac{b}{2}\xi_{xx}. \tag{4.8}$$

The solution of this equation depends on the values of b and d:

4.1. Case $b \neq 0, d \neq 0$. Substituting β into the second equation (4.3), one gets

$$\xi\xi_{xx} - \frac{\xi_x^2}{2} + 2c\xi^2 = C_3, \tag{4.9}$$

C_3 is an arbitrary constant.

If $C_2 \neq 0$, then from equations (4.7), (4.8) and (4.9), one obtains

$$\xi = \frac{C_2}{b},\ \eta = y\left(\frac{C_2}{2}(\frac{1}{b})' + C_1\right) + \gamma,$$
$$c = \frac{1}{2}\left[C_5 b^2 - \frac{3}{2}(\frac{b'}{b})^2 + \frac{b''}{2b}\right],\ d = \frac{b'}{2} + C_4 b^2,$$

where C_4 is an arbitrary constants, $C_5 = C_3/C_2$, and $\gamma(x)$ is an arbitrary solution of (3.1). Since $\xi = \xi_\tau$, the the coefficient b has to satisfy the same property $b = b_\tau$. The infinitesimal generator obtained is

$$X = C_1 y\partial_y + C_2\left(\frac{1}{b}\partial_x + \frac{y}{2}\left(\frac{1}{b}\right)'\partial_y\right) + \gamma\partial_y. \tag{4.10}$$

If $C_2 = 0$, then $\xi = 0,\ \eta = C_1 y + \gamma$ and all coefficients are arbitrary. The infinitesimal generator is

$$X = (C_1 y + \gamma)\partial_y. \tag{4.11}$$

4.2. Case $b \neq 0, d = 0$. Solving equations (4.6), (4.7), (4.8) and the second equation of (4.3), one obtains $\beta_\tau = C_6$, $b\xi = C_2$, $\xi = C_7 x + C_8$, $c\xi^2 = C_9$, where C_6, C_7, C_8, C_9 are arbitrary constants. Since $\xi = \xi_\tau$, then $C_7 = 0$.

If $C_8 \neq 0$, then

$$c = \frac{C_9}{C_8^2},\ b = \frac{C_2}{C_8}. \tag{4.12}$$

The infinitesimal generator of the admitted Lie group is

$$X = C_8\partial_x + (C_6 y + \gamma)\partial_y. \tag{4.13}$$

If $C_8 = 0$, then $\xi = 0,\ \eta = C_6 y + \gamma$, b and c are arbitrary, $\gamma(x)$ is an arbitrary solution of (3.1). The infinitesimal generator is

$$X = (C_6 y + \gamma)\partial_y.$$

4.3. Case $b = 0, d \neq 0$. From equation (4.8), one finds $d\xi^2 = C_{10}$, where C_{10} is an arbitrary constant. Hence,

$$\xi = \left(\frac{C_{10}}{d}\right)^{1/2},\ \eta = -\left(\frac{C_{10}}{4}\frac{d'}{d^{3/2}} + C_1\right)y + \gamma. \tag{4.14}$$

If $C_{10} \neq 0$, then from equation (4.9) one finds

$$c = \frac{1}{2}\left[\frac{C_3}{C_{10}}d + \frac{d'}{2d} + \frac{1}{8}(\frac{d'}{d^2})^2\right]. \tag{4.15}$$

The infinitesimal generator obtained is

$$X = \frac{C_{10}}{d^{1/2}}\partial_x + \left(-\frac{C_{10}^{1/2}d'}{2d^{3/2}} + C_1 + \gamma\right)\partial_y. \tag{4.16}$$

If $C_{10} = 0$, then $\xi = 0, \beta = C_1, \eta = C_1 y + \gamma$ and the coefficients c and d are arbitrary functions. Hence, the infinitesimal generator is

$$X = (C_1 y + \gamma)\partial_y. \tag{4.17}$$

5. Conclusion

The linear second-order delay ordinary differential equation is classified into three cases as the followings:

- $b \neq 0$ and $d \neq 0$. The infinitesimal generator admitted by the equation of this case is (4.10)
- $b \neq 0$ and $d = 0$. The infinitesimal generator admitted by the equation of this case is (4.13)
- $b = 0$ and $d \neq 0$. The infinitesimal generator admitted by the equation of this case is (4.16).

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