

Group Classification of Linear Second-Order Delay Ordinary Differential Equation

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Outline

- 1 Motivation
 - History and Previous Researches
 - Group Analysis /Basic Concept
- 2 Problems
 - Linear Second-Order Delay Differential Equation
- 3 Research Procedure
 - Constructing/Solving Determining Equation
- 4 Main results
 - Equivalence Symmetry of Second-Order Delay ODE
 - Group Classification of Second-Order Delay ODE

History and previous researches.

- In 1870, S. Lie first introduced Group analysis.
- In 1960, L.V. Ovsiannikov establishing modern group analysis
- In 1996, N.H. Ibragimov is editor of *Lie Group Analysis of Differential Equations* Vol.1-3
- In 2002, J. Tanthanuch and S.V. Meleshko, *Application of Group Analysis to Delay Differential equation.*

Group Analysis / Symmetry Methods.

- Let us consider ordinary differential equation

$$F(x, y, y', y'', \phi) = 0. \quad (1)$$

- Consider invertible transformation

$$\bar{x} = \varphi^x(x, y, \phi(x, y), \epsilon)$$

$$\bar{y} = \varphi^y(x, y, \phi(x, y), \epsilon)$$

$$\bar{\phi} = \varphi^\phi(x, y, \phi(x, y), \epsilon)$$

with real parameter ϵ , which $\bar{x} = x, \bar{y} = y, \bar{\phi} = \phi$, when $\epsilon = 0$.

Lie group of transformation

- Suppose that the transformations map a solution of a given equation to a solution of the equation which has the same structure.

$$F(x, y, y', y'', \phi) = 0. \Rightarrow F(\bar{x}, \bar{y}, \bar{y}', \bar{y}'', \bar{\phi}) = 0.$$

- We call these transformation “**symmetries**”.
- The set of these symmetries forms *a local one-parameter Lie group* or *Lie group*.

Infinitesimal Generator.

- Lie theorem constructs a **one to one** correspondence between the Lie group of transformations and the infinitesimal generator X which is defined as the following:

$$X = \xi \partial_x + \eta \partial_y + \zeta \partial_\phi, \quad (2)$$

where

$$\xi(x, y, \phi) = \left. \frac{\partial \varphi^x}{\partial \epsilon} \right|_{\epsilon=0},$$
$$\eta(x, y, \phi) = \left. \frac{\partial \varphi^y}{\partial \epsilon} \right|_{\epsilon=0},$$
$$\zeta(x, y, \phi) = \left. \frac{\partial \varphi^\phi}{\partial \epsilon} \right|_{\epsilon=0}.$$

- *Bäcklund canonical infinitesimal generator*, \tilde{X} is defined by

$$\tilde{X} = (\eta - y'\xi)\partial_y + \eta^{y'}\partial_{y'} + \eta^{y''}\partial_{y''} + (\zeta - \phi_x\xi)\partial_\phi + \zeta^{\phi_y}\partial_{\phi_y} \quad (3)$$

where

$$\xi(x, y, \phi) = \frac{\partial\varphi^x}{\partial\epsilon}(x, y, \phi; 0),$$

$$\eta(x, y, \phi) = \frac{\partial\varphi^y}{\partial\epsilon}(x, y, \phi; 0),$$

$$\zeta(x, y, \phi) = \frac{\partial\varphi^\phi}{\partial\epsilon}(x, y, \phi; 0),$$

$$\eta^{y'} = D_x(\eta - y_x\xi), \quad \eta^{y''} = D_x(\eta_{y'}), \quad \zeta^{\phi_y} = D_y(\zeta - \phi_x\xi),$$

$$D_y = \partial_y + \phi_y\partial_\phi + \phi_{xy}\partial_{\phi_x} + \dots,$$

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \dots + (\phi_x + y'\phi_y)\partial_\phi + (\phi_{xx} + y'\phi_{xy})\partial_{\phi_x}$$

Determining equation

- Both of them are equivalent,

$$\tilde{X} \sim X.$$

- A Lie group of transformations is *admitted* by the equation if it satisfies

$$\tilde{X}F(x, y, y', y'', \phi) \Big|_{F(x, y, y', y'', \phi)=0} = 0. \quad (4)$$

- Equation (4) is called *a determining equation*.

Splitting determining equation

- By existence theory of an initial value problem guarantees the existence of solution $y(x)$. Therefore, by the arbitrariness of x_0 and initial function y_0 , the determining equation can be split into a system of several equations and can be solved analytically.
- Unknown functions ξ, η and ζ can be obtained from this process.

Linear Second-Order Delay Differential Equation

- Consider a linear second-order delay ordinary equation,

$$y''(x) + a(x)y'(x) + b(x)y'(x - \tau) + c(x)y(x) + d(x)y(x - \tau) = g(x).$$

Denote $y_\tau = y(x - \tau)$,
 $y'_\tau = y'(x - \tau)$.

- The equation is simply written as

$$y'' + a(x)y' + b(x)y'_\tau + c(x)y + d(x)y_\tau = g(x). \quad (5)$$

- Considering the change $\tilde{x} = x$, $\tilde{y} = y - y_p$, where y_p is a particular solution of equation (5). Equation (5) is reduced to a homogeneous equation

$$y'' + a(x)y' + b(x)y'_\tau + c(x)y + d(x)y_\tau = 0.$$

- Coefficient $a(x)$ can be reduced by letting $y = v(x)q(x)$ with q satisfying $2q' + aq = 0$,

$$y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau = 0. \quad (6)$$

- We will consider equivalence symmetries of equation (6) instead of (5).

Constructing / Splitting determining equation

- Let $F = y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau$, then the determining equation is

$$\tilde{X} (y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau) \Big|_{F=0} = 0. \quad (7)$$

Determining equation

$$\begin{aligned}
 & 2\eta_{bc}b'c' + 2\eta_{bd}b'd' + 2\eta_{bx}b' + 2\eta_{by}b'y' + \eta_{bb}b'^2 + \eta_b b'' \\
 & + 2\eta_{cd}c'd' + 2\eta_{cx}c' + 2\eta_{cy}c'y' + \eta_{cc}c'^2 + \eta_c c'' + 2\eta_{dx}d' \\
 & + 2\eta_{dy}d'y' + \eta_{dd}d'^2 + \eta_d d'' + 2\eta_{xy}y' + \eta_{xx} + \eta_{yy}y'^2 - \eta_y by'_\tau \\
 & - \eta_y cy - \eta_y dy_\tau + \eta_b^\tau bb'_d + \eta_c^\tau bc'_d + \eta_d^\tau bd'_d + \eta_x^\tau b + \eta_y^\tau by'_\tau \\
 & - 2\xi_{bc}b'c'y' - 2\xi_{bd}b'd'y' - 2\xi_{bx}b'y' - \xi_{bb}b'^2y' + 2\xi_b bb'y'_\tau \\
 & + 2\xi_b b'cy + 2\xi_b b'dy_\tau - \xi_b b''y' - 2\xi_{cd}c'd'y' - 2\xi_{cx}c'y' \\
 & - \xi_{cc}c'^2y' + 2\xi_c bc'y'_\tau + 2\xi_c cc'y + 2\xi_c c'dy_\tau - \xi_c c''y' \\
 & - 2\xi_{dx}d'y' - \xi_{dd}d'^2y' + 2\xi_d bd'y'_\tau + 2\xi_d cd'y + 2\xi_d dd'y_\tau \\
 & - \xi_d d''y' - \xi_{xx}y' + 2\xi_x by'_\tau + 2\xi_x cy + 2\xi_x dy_\tau - \xi_b^\tau bb'_\tau y'_\tau \\
 & - \xi_c^\tau bc'_\tau y'_\tau - \xi_d^\tau bd'_\tau y'_\tau - \xi_x^\tau by'_\tau - bb_\tau \xi y'_{2\tau} + bb_\tau \xi^\tau y'_{2\tau} - bc_\tau \xi y_\tau
 \end{aligned}$$

Determining equation (continued)

$$+bc_{\tau}\xi^{\tau}y_{\tau} - bd_{\tau}\xi y_{2\tau} + bd_{\tau}\xi^{\tau}y_{2\tau} + c\eta + d\eta^{\tau} + d\xi y'_{\tau} \\ - d\xi^{\tau}y'_{\tau} + y\zeta^c + y_{\tau}\zeta^d + y'_{\tau}\zeta^b = 0.$$

- Splitting this equation with respect to $y', y'_{\tau}, y'_{2\tau}, b'', c'', d'', b', c', d'$ and later with respect to y, y_{τ} , one finds

$$\xi = \xi_{\tau} = \alpha, \eta = \beta y + \gamma, \zeta^b = b(-\alpha' + \beta - \beta_{\tau}), \\ \zeta^c = -\beta'' - 2c\alpha', \zeta^d = -b\beta'_{\tau} - 2d\alpha' + d\alpha - d\alpha_{\tau},$$

where $\alpha(x), \beta(x)$ are arbitrary periodic functions with the periodic τ , $\gamma(x)$ is an arbitrary solution of (6)

The equivalence symmetry of second-order delay ordinary differential equation.

- Thus the equivalence symmetry is

$$\bar{x} = \alpha(x), \bar{y} = \beta(x)y + \gamma(x), \quad (8)$$

where $\alpha(x), \beta(x)$ are arbitrary periodic functions with the periodic τ , $\gamma(x)$ is an arbitrary solution of (6)

Group Classification of Second-Order Delay ODE

- Lie group of transformations admitted by equation (6) has to satisfy the following determining equation

$$\tilde{X}(y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau) \Big|_{(6)} = 0. \quad (9)$$

- The operator \tilde{X} is defined by

$$\tilde{X} = (\eta - y'\xi)\partial_y + (\eta_\tau - y'_\tau\xi_\tau)\partial_{y_\tau} + \eta^{y'}\partial_{y'} + \eta^{y'_\tau}\partial_{y'_\tau} + \eta^{y''}\partial_{y''}.$$

where

$$\xi(x, y) = \frac{\partial\varphi^x}{\partial\epsilon}(x, y; 0), \quad \eta(x, y) = \frac{\partial\varphi^y}{\partial\epsilon}(x, y; 0),$$

$$\xi_\tau(x, y) = \xi(x - \tau, y_\tau), \quad \eta_\tau(x, y) = \eta(x - \tau, y_\tau)$$

$$\eta^{y'} = D_x(\eta - y'\xi), \quad \eta^{y'_\tau} = D_x(\eta_\tau - y'_\tau\xi_\tau), \quad \eta^{y''} = D_x(\eta^{y'}),$$

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots + y'_\tau\partial_{y_\tau} + y''_\tau\partial_{y'_\tau}.$$

Split determining equation

- Splitting this equation with respect to y' , y'_τ , $y'_{2\tau}$ and later with respect to y , y_τ , one finds

$$\xi = \xi_\tau, \quad \eta = \beta y + \gamma, \quad (10)$$

$$\xi_{xx} = 2\beta', \quad \beta'' = -c'\xi - 2c\xi_x, \quad (11)$$

$$\gamma'' = -b\gamma'_\tau - c\gamma - d\gamma_\tau, \quad (12)$$

$$b(\beta - \beta_\tau) = b'\xi + \xi_x b, \quad (13)$$

$$d(\beta - \beta_\tau) = d'\xi + b\beta'_\tau + 2\xi_x d. \quad (14)$$

- By integrating (11), one finds $\beta = \xi_x/2 + C_1$. Since $\xi = \xi_\tau$, it implies $\beta = \beta_\tau$. Hence, integrating equation (13) one has

$$b\xi = C_2. \quad (15)$$

Equation (14) is written

$$d'\xi + 2\xi_x d = -\frac{b}{2}\xi_{xx}. \quad (16)$$

- Solving this equation with respect to ξ depends on values of b and d one gets

- case $b \neq 0, d \neq 0$

- The obtained infinitesimal generator is

$$X = C_1 y \partial_y + C_2 \left(\frac{1}{b} \partial_x + \frac{y}{2} \left(\frac{1}{b} \right)' \partial_y \right) + \gamma \partial_y. \quad (17)$$

- Case $b \neq 0, d = 0$

- The infinitesimal generator is of admitted Lie group is

$$X = C_8 \partial_x + (C_6 y + \gamma) \partial_y. \quad (18)$$

- Case $b = 0, d \neq 0$

- The obtained infinitesimal generator is

$$X = \frac{C_{10}}{d^{1/2}} \partial_x + \left(-\frac{C_{10}^{1/2} d'}{2d^{3/2}} + C_1 + \gamma \right) \partial_y \quad (19)$$

Thank you for you attention!

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