ENGINEERING CALCULUS II
MTH 1212

Polar Coordinates

by

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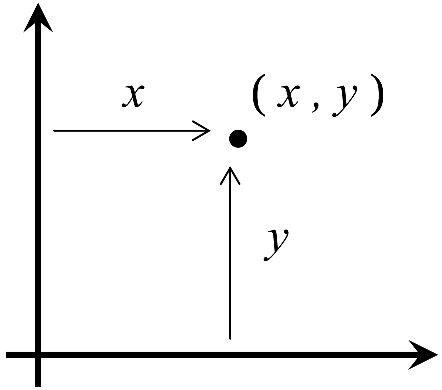
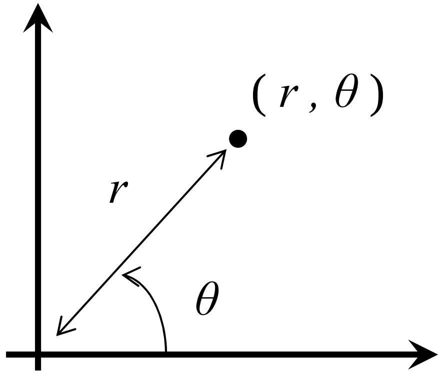
Polar coordinates

Representation of coordinates by parameter r and θ instead of x and y

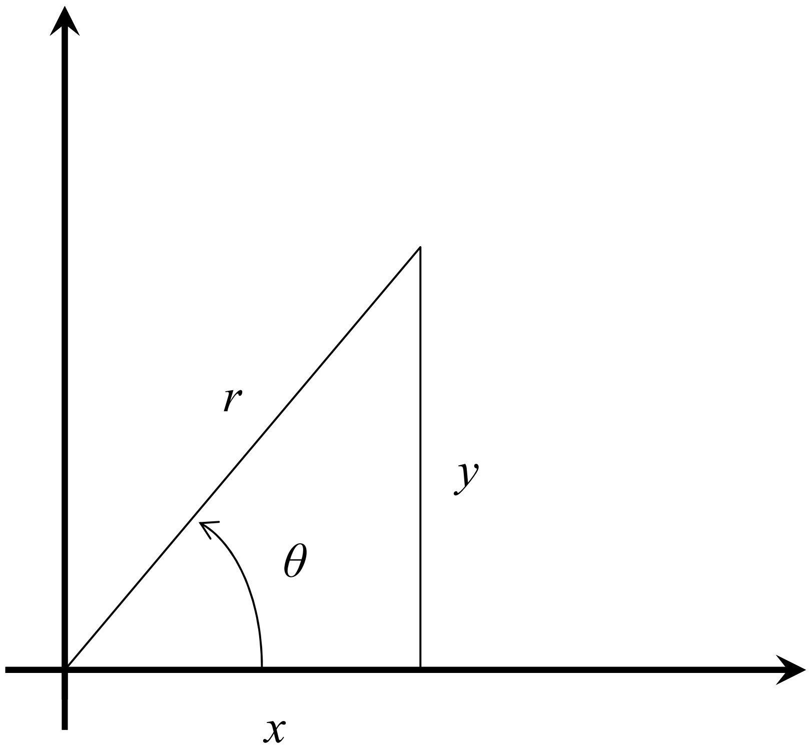


Cartesian representation



Polar representation

Relationship between Cartesian & Polar



$$(x, y) \xrightarrow{\text{cartesian to polar}} (r, \theta)$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$(r, \theta) \xrightarrow{\text{polar to cartesian}} (x, y)$$

$$x = r\cos\theta$$

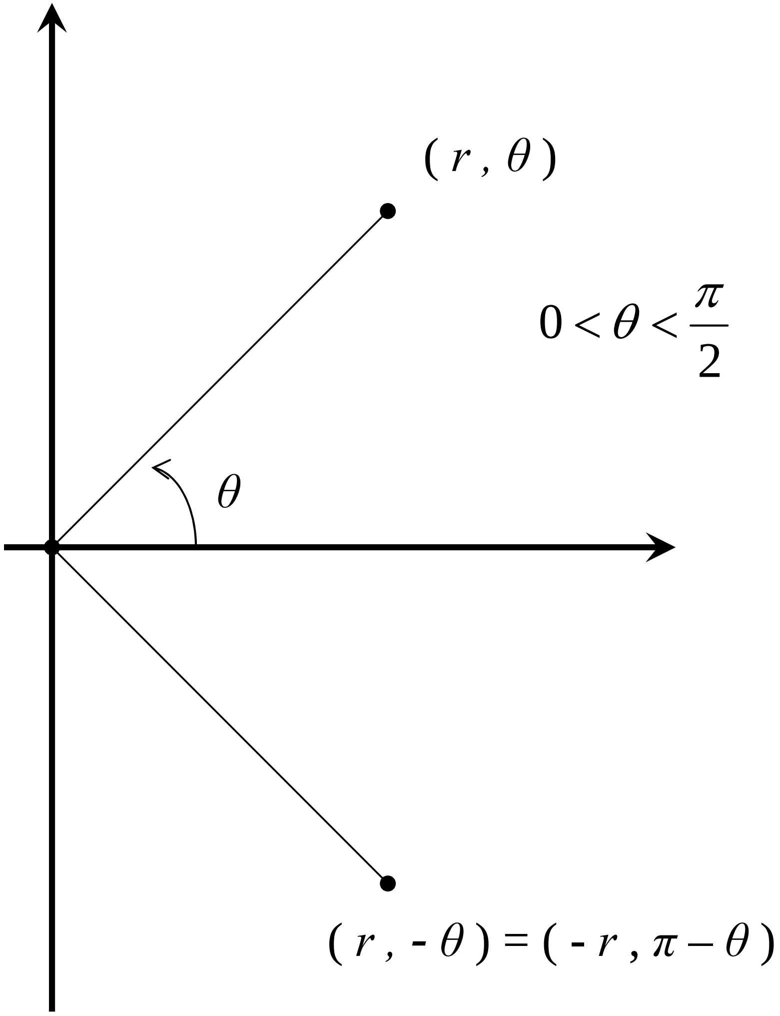
$$y = r\sin\theta$$

Symmetry – about x-axis

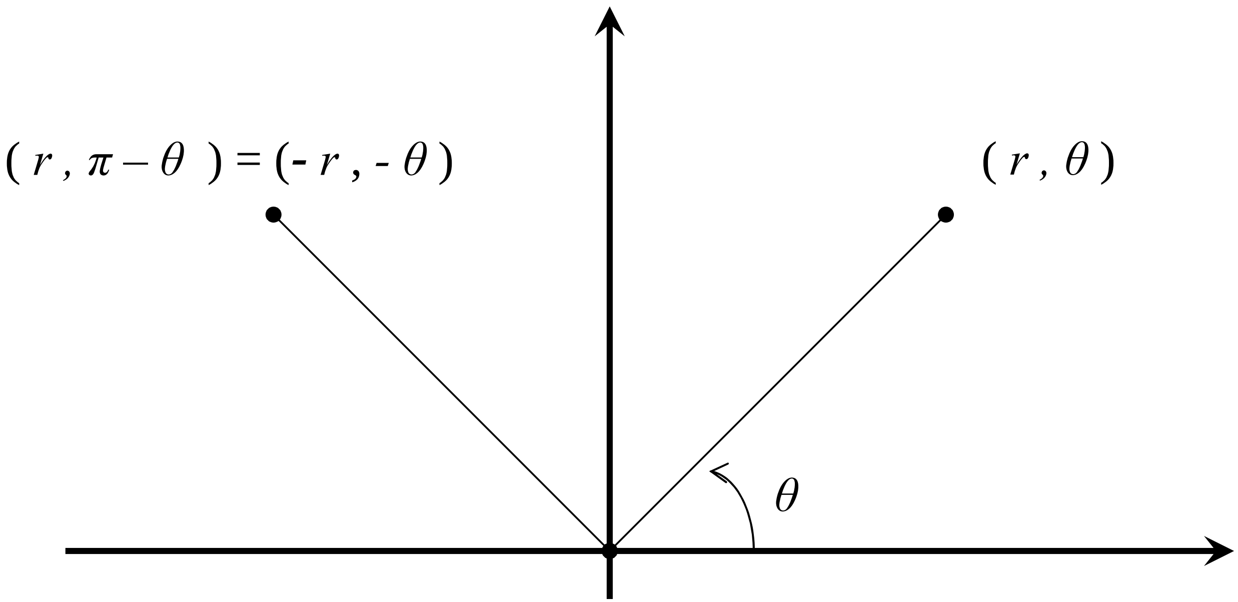
$$x = r\cos(2\pi - \theta)$$
$$= -r\cos(-\theta)$$

$$y = r\sin(2\pi - \theta)$$
$$= -r\sin\theta = r(-\sin\theta) = r\sin(-\theta)$$

$$\cos(2\pi - \theta) = \cos\theta$$
$$\sin(2\pi - \theta) = -\sin\theta$$



Symmetry – about y-axis



$$0 < \theta < \frac{\pi}{2}$$

$$x = r\cos(\pi - \theta)$$
$$= -r\cos\theta = -r\cos(-\theta)$$

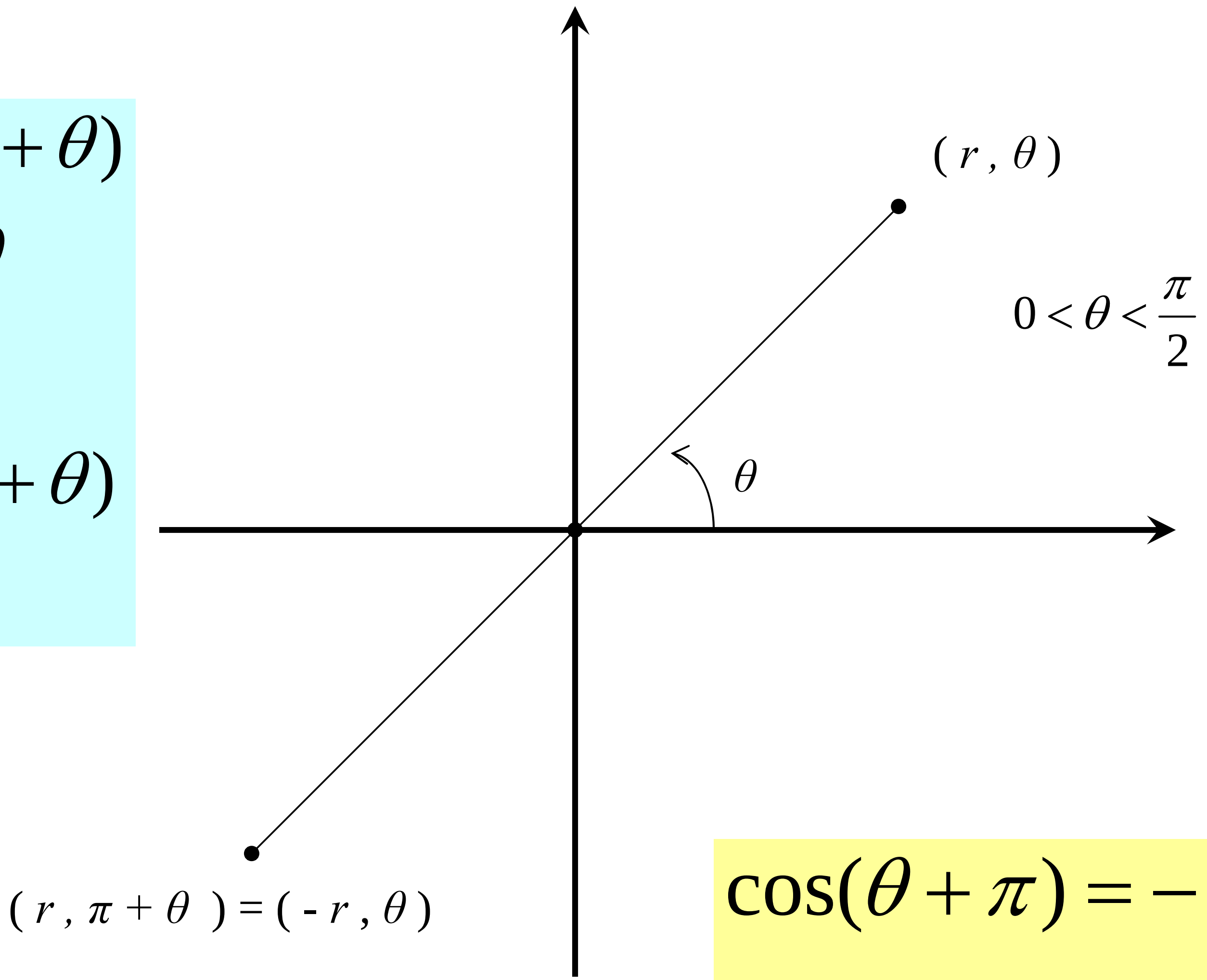
$$y = r\sin(\pi - \theta)$$
$$= r\sin\theta = (-r)(-\sin\theta) = -r\sin(-\theta)$$

$$\cos(\theta - \pi) = -\cos\theta$$
$$\sin(\theta - \pi) = \sin\theta$$

Symmetry – about the origin

$$x = r\cos(\pi + \theta)$$
$$= -r\cos\theta$$

$$y = r\sin(\pi + \theta)$$
$$= -r\sin\theta$$



$$\cos(\theta + \pi) = -\cos\theta$$
$$\sin(\theta + \pi) = -\sin\theta$$

Calculus of polar coordinates – Differentiation

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Calculus of polar coordinates – Integration

Curve length

$$ds^2 = dx^2 + dy^2$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$s = \int_{\theta_0}^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + r^2\cos^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta + r^2\sin^2\theta - 2r\frac{dr}{d\theta}\sin\theta\cos\theta$$

$$s = \int_{\theta_0}^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

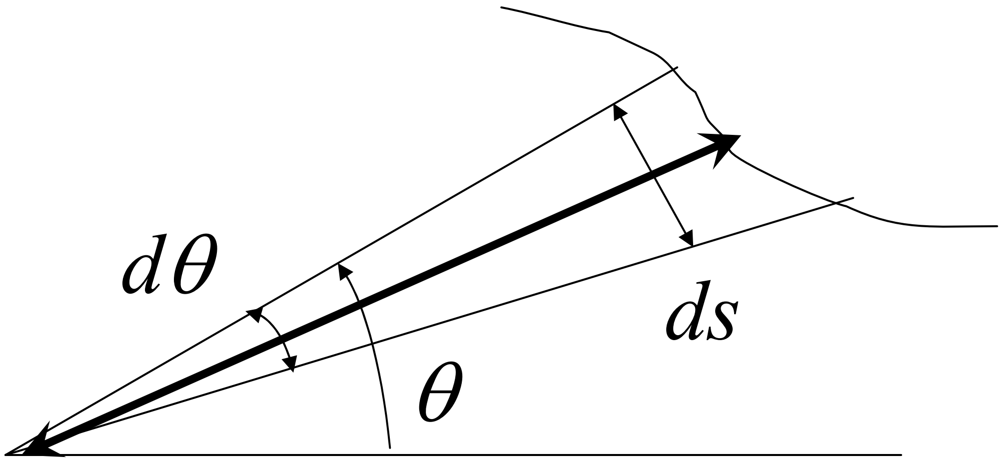
$$\boxed{s = \int_{\theta_0}^{\theta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta}$$

Calculus of polar coordinates – Integration

Sector area

$$dA = \frac{1}{2} r ds \qquad (1)$$

$$ds = r d\theta \qquad (2)$$



$$dA = \frac{1}{2} r^2 d\theta$$

$$A = \int \frac{1}{2} r^2 d\theta$$