Sketching Graphs

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1 Functions

Most often, a function $y = f(x)$ is required to be plotted.

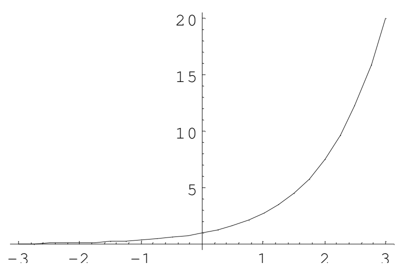
Typically, it is needed to find out the domain of definition of the function, the intercepts, the extrema, the critical points, regions where the function is increasing (i.e. y' is positive), where it is decreasing, and the asymptotes in order to finally draw the graph. This is how you should approach while drawing a graph as an answer to the subjective paper.

A horizontal or vertical asymptote can be detected as usual by checking the limits. An asymptote of the form $y = ax + b$ can be detected by testing whether $a = \lim_{x \to \pm\infty} \frac{f(x)}{x}$ is finite, and if so, whether $b = \lim_{x \to \pm\infty} \left[f(x) - ax \right]$ is finite.

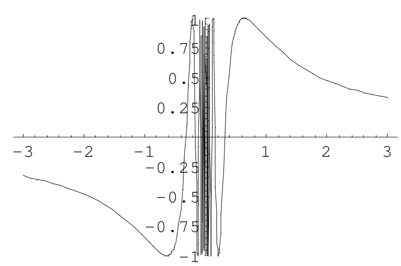
Exercise 1: Draw the graph of $y = \frac{x^2 + 1}{x - 1}$.

Below are some examples.

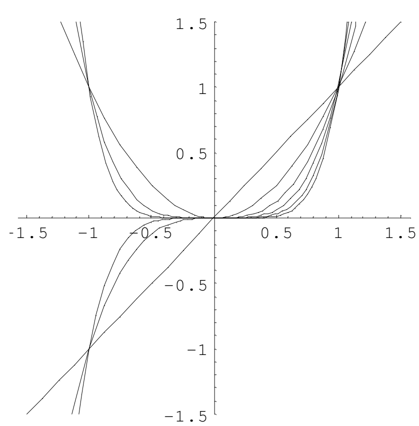
Exponential: $y = e^x$



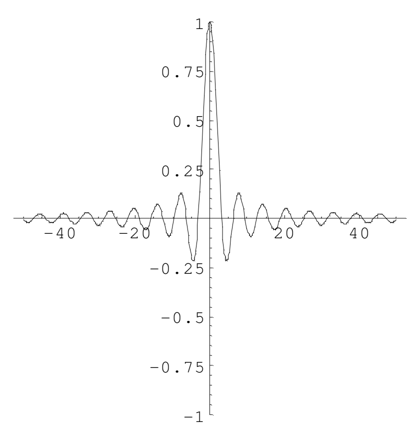
$y = \sin(1/x)$



Powers: $y = x^n$, for n=1,2,3,4,5, and 6



$y = \sin(x)/x$

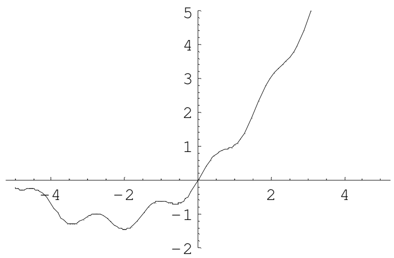


Exercise 2: Draw the graph of y = ln(x)/x.

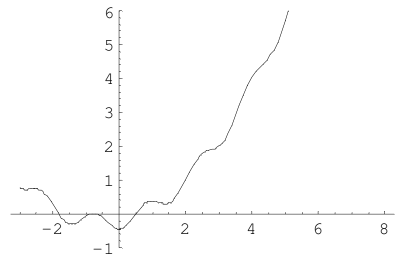
1.1 Location Transformations

The graph of $y - b = f(x - a)$ is the graph of $y = f(x)$ shifted by 'b' units to the right and 'a' units upwards. Of course, the direction of these movements will be opposite when either 'a' or 'b' is negative.

$y = f(x)$



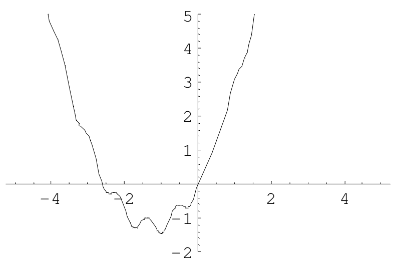
$y - 1 = f(x - 2)$



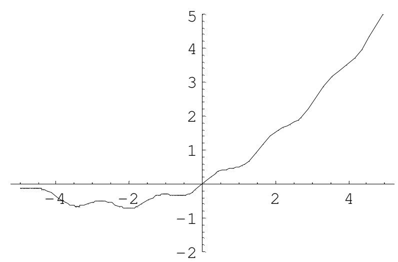
1.2 Scale Transformations

If both 'a' and 'b' are both greater than 1, the graph of $by = f(ax)$ is same as the graph of $y = f(x)$ shrunk in both directions, by the corresponding constants. If any of 'a' or 'b' is less than one, there is an expansion in that direction instead.

$y = f(2x)$



$2y = f(x)$

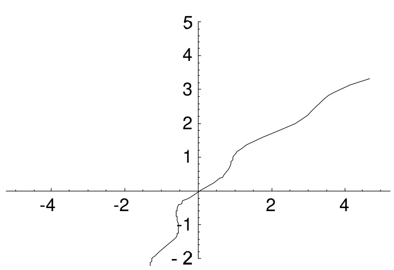


1.3 Reflections and Rotations

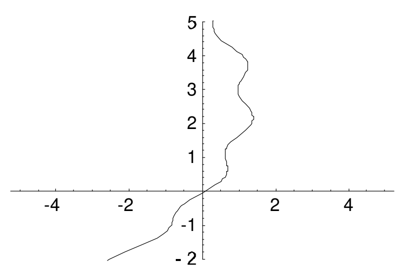
The reflections along the x axis and y axis are obtained for graphs of $-y = f(x)$ and $y = f(-x)$ respectively.

The following two reflections (along the lines $y = x$, and $y = -x$) are also important.

$x = f(y)$



$-x = f(-y)$



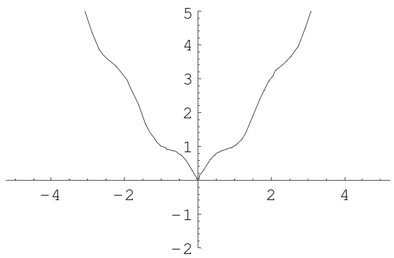
The graph gets rotated counter-clockwise by 90° when plotting $x = f(-y)$,

⇒ the graph gets rotated by 180° when plotting $-y = f(-x)$,

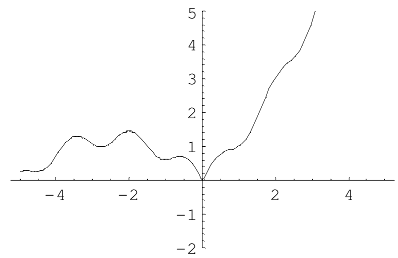
and the graph gets rotated clockwise by 90° when plotting $-x = f(y)$.

1.4 Other Transformations

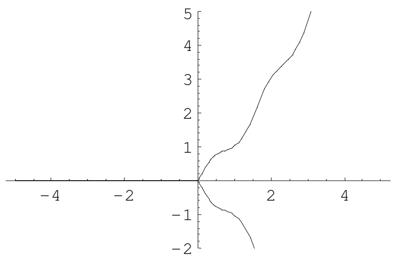
$y = f(|x|)$



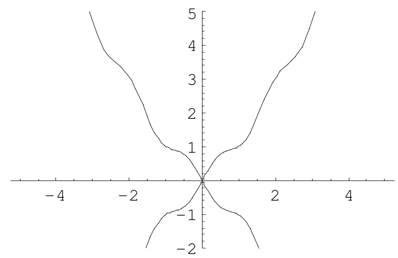
$y = |f(x)|$



$|y| = f(x)$



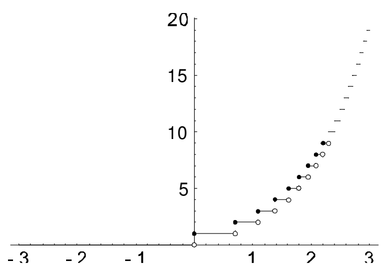
$|y| = f(|x|)$



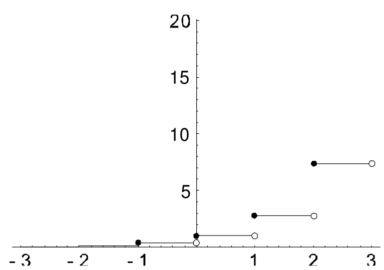
Exercise 3: Draw the graphs of $|y| = -f(x)$, $|y| = -f(|x|)$ and $|y| = -|f(x)|$.

In the following, '$[x]$' denotes the highest integer not exceeding x, and 'Frac(x)' denotes $x - [x]$.

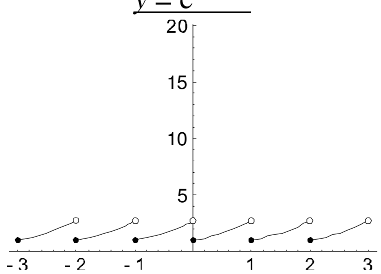
$y = [e^x]$



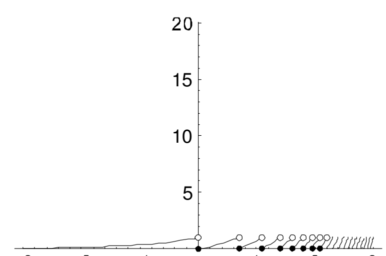
$y = e^{[x]}$



$y = e^{\mathrm{Frac}(x)}$



$y = \mathrm{Frac}(e^x)$



Compare the above graphs with graph of e^x on page 1 to get a general idea of how these transformations work.

Exercise 4: Plot $y = [2.5\sin(x)]/2.5$

Exercise 5: Plot $y = |2 \times \mathrm{Frac}(x/2) - 1|$

Final Exercise 1: Using the result of Exercise 2, draw a rough sketch of $x^y = y^x$.

2 Relations

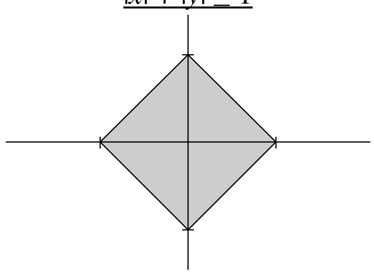
A relation **R** on real numbers is a function from $\mathbb{R}^2$ to {T, F}. The graph of the relation is defined as all the points in Cartesian plane whose coordinates map to T, i.e. the points (x, y) : $\mathbf{R}(x, y) = \mathrm{T}$. Instead of writing '$\mathbf{R}(x, y) = \mathrm{T}$', it is also often written '$x\ \mathbf{R}\ y$' (e.g. '$x \geq y$' when $\mathbf{R} \equiv \geq$). The functions discussed in the previous section also fall under this category, as for any function f, graph of the relation **R** defined as $x\ \mathbf{R}\ y \Leftrightarrow y = f(x)$ will give the graph of that function. So whatever can be said about plotting relations goes through for plotting functions also. Remarkably, most facts for graphs of functions also very easily extend to relations.

If the relation is $y \leq f(x)$ ($y \geq f(x)$), the trick is to plot $y = f(x)$ and then for each partition of $\mathbb{R}^2$ formed by the line choose any point (a, b), shade that partition only if $b \leq f(a)$ ($b \geq f(a)$).

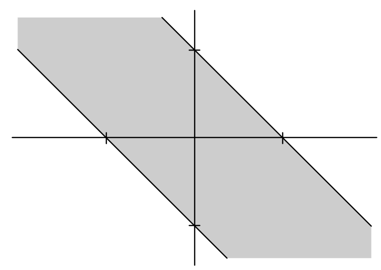
Exercise 6: Shade the region given by $x \leq y^2$.

In general relations can be plotted by examining the set of values $\{y: x\ \mathbf{R}\ y\}$ for each $x \in \mathbb{R}$. It may be noted that the graph $\{(x, y): x\ \mathbf{R}\ y\}$ may not always be a closed set, so to avoid problems in sketching it we will generally sketch the closure of this set. Below are some examples.

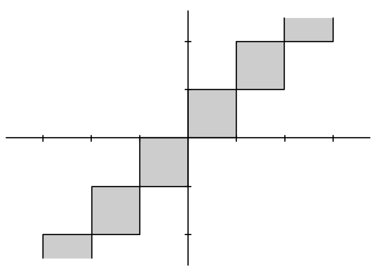
$|x| + |y| \leq 1$



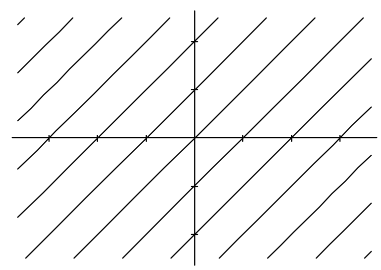
$|x + y| \leq 1$



$[x] = [y]$



$\mathrm{Frac}(x) = \mathrm{Frac}(y)$



Note that the top and right boundaries of the squares in $[x] = [y]$ would not belong to the graph unless we considered the closure.

Exercise 7: Show graphically that $|x + y| \leq 1 \cap |x - y| \leq 1 \Leftrightarrow |x| + |y| \leq 1$.

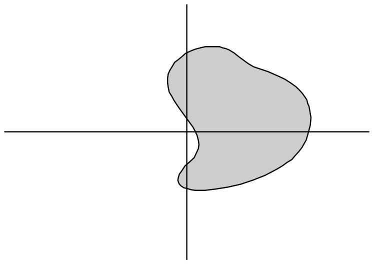
Exercise 8: How will the graph of $[x] = [y] = [z]$ look like in 3-Dimensions?

Exercise 9: What would be the graph of $\mathrm{Frac}(x + y) = \mathrm{Frac}(x - y)$?

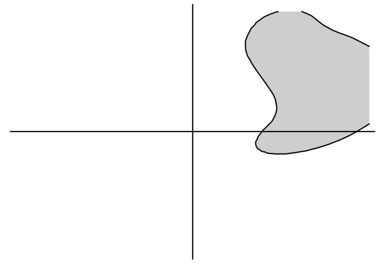
2.1 Transformations

Results for almost all the transformations discussed in the previous section can be generalized for relations as well. Some examples are given below.

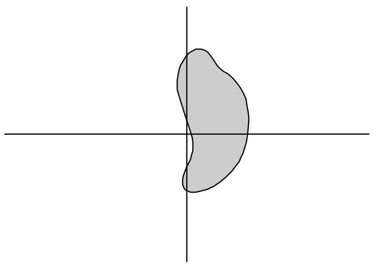
x **R** y



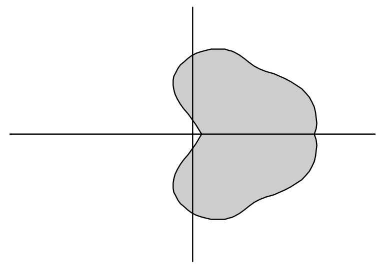
$x - 2$ **R** $y - 1$



$2x$ **R** y



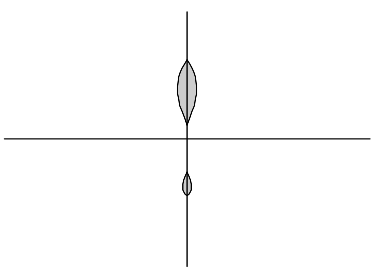
x **R** $|y|$

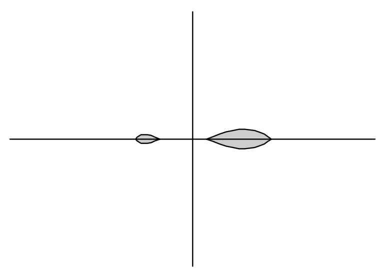


Exercise 10: Think about the graphs of the above relation under any of the other transformations you can think of in analogy with the previous section.

Exercise 11: If **R** is the above relation, shade the region $|x|$ **R** $-|y|$.

Exercise 12: Identify the following graphs on basis of the above **R**.





Exercise 13: Draw the graph of $1 - |x| \geq y \geq (|x| - 1)^2 - 4$.

Final Exercise 2: Draw the graph of $\left[\frac{y-x}{\sqrt{2}}\right] = \left[\frac{y+x}{\sqrt{2}}\right]$. Can you identify it with a rotation of any of the graphs you have seen here?