Some Interesting Problems

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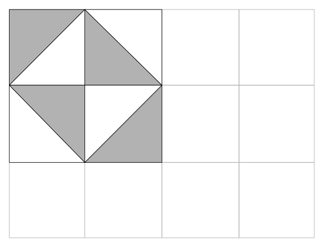
This is a collection of some interesting problems in Mathematics, many of which are fairly non-trivial.

If you get stuck and really want solution to any of the problems you can mail me!

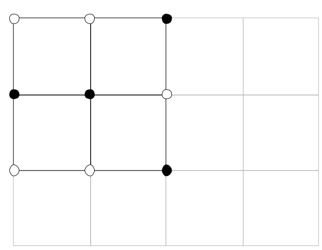
Also visit my site at http://www.geocities.com/hirak_99.

Problem 1. A 12×12 board can be covered without overlaps by tiles of size 3×1 so that the tiles take up no extra space. If one corner of the board is removed, this is no longer possible since $12^2 - 1$ is not divisible by 3. Show that even if 3 corners were removed, this would not be possible.

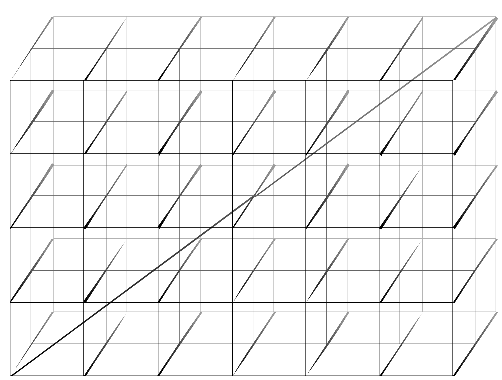
Problem 2. In an $m \times n$ chess board, suppose the squares are to be covered with mn black triangles and mn white triangles all of same size, which is half of the squares. In how many ways can the chess board be covered such that no black triangle is adjacent to another black triangle, and no white triangle is adjacent to another white triangle?



Problem 3. In an $m \times n$ grid, there are $(m+1)(n+1)$ lattice points. Each square is surrounded by four of them. The task is to put blue beads and red beads in all the lattice points such that each of the squares has exactly two blue and two red beads surrounding it. In how many ways can it be accomplished?



Problem 4. If a straight line is drawn from (0, 0, 0) to coordinates (a, b, c), where a, b, c are positive integers, how many unit cubes with integral vertices will it pass through?



Problem 5. Teams of 100 people are playing a game. The 100 people from each team will be made to stand in a row. The last person can see the 99 people before him, 2nd last person can see the first 98 people, and so on. Each of them is then given to wear a hat, which is randomly coloured as either black or white. From the last person to the first person, each of them is then asked which colour he is wearing. They can only answer 'Black', or 'White'. The team scores

1 point for each correct answer, and 0 for incorrect. Each team can fix any strategy before attempting the game. What is the maximum score guaranteed? What would be the maximum score guaranteed if there were a thousand colours for the hats instead of two?

Problem 6. Show there are exactly $\frac{1}{n+1}\binom{2n}{n}$ many possible binary trees with n nodes.

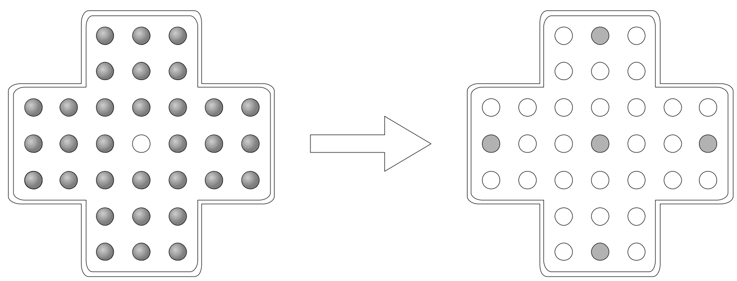
Problem 7. Let m and n be two positive integers, n being a prime other than 2 or 5, and m not a multiple of n. When m is divided by n, if there are even number of digits in the recurring part of the decimal number thus formed, show that the average of these digits is 4.5. Also show that if there are an odd number of digits, then the average is necessarily different from 4.5.

Example:

$14/61=0.\overline{229508196721311475409836065573770491803278688524590163934426}$,

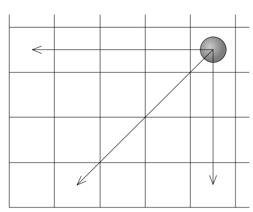
comprising of 60 recurring digits, the average of these digits is 4.5. What this means is that the 'large digits' and 'small digits' occur with equal frequency in such decimals.

Problem 8. Show that in the game of Solitaire (Brainvita), if the game is solved and there is only one bead remaining, it must be in one of the five positions indicated below.



Problem 9. How many cubes can be formed by joining any eight vertices of any of the 8^3 unit cubes, which are arranged in space to form an $8 \times 8 \times 8$ large cube? (Note that the faces of the cubes which can be formed in this fashion may not be parallel to the faces of the unit cubes.)

Problem 10. A game is to be played by two players on a board which extends indefinitely upwards and to the right. First a neutral person (the computer) chooses a random square, and places a coin on it. From then on each person takes turn to move the coin *any* number of squares to the left, or downwards, or following the lower left diagonal as shown in the figure. The first player to get it to the lower left square so that the second player can't move any more wins. Find an algorithm or write a program to win the game whenever a win is possible.



Problem 11. In a 3-dimensional cube (i.e. an ordinary cube), there are twelve 1-dimensional faces (i.e. edges), and six 2-dimensional faces (i.e. sides), and one 3-dimensional face (i.e. the whole cube). Show that in general the number of r-dimensional faces/sides of an n-dimensional cube is given by $\binom{n}{r}2^{n-r}$.

Problem 12. For the fifteen's puzzle, half of the configurations (i.e. $16!/2$ configurations) are reachable and the other half are unreachable starting from a solved position. How many combinations of a Rubik's Cube are reachable from a solved Cube using the 'allowed' moves? How many are unreachable ('impossible')?

Problem 13. Two players are playing a game. There are an odd number of sticks on the table initially. Each player takes turn to pick up exactly 1, 2, or 3 sticks. When all the sticks are picked up, the player having odd number of sticks wins the game. Decide for which n a win is possible for the first player (however brilliant the opponent is). Hence find a strategy for the game.

Problem 14. There are n switches and n lights. The i^{th} switch changes the state (on or off) of i^{th} light. If i^{th} switch changes the state of j^{th} light, then j^{th} switch changes the state of i^{th} light. If initially all the lights are off, based on the above information show that it is possible to switch on all the lights simultaneously.

Problem 15. In an n dimensional tic-tac-toe grid, how many winning lines are there which pass through the centre? (For example there are only 2 winning lines passing thru the centre for a regular 2-dimensional grid.)