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$so(2,1)$ scattering via Euclidean Connection for the Natanzon Hypergeometric Potentials

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Abstract

The $so(2,1)$ Euclidean Connection [1] formalism is used to calculate the S matrix for the Hypergeometric Natanzon Potentials (HNP) [2].

1 Introduction

Algebraic scattering has been an active research field for many years. Pioneer works in the late 1960's and 1970's by Zwanziger [3] and Barut, et al. [4] dealing with the Coulomb system founded the basic ideas of this area. More recently, Alhassid et al. [1] developed an algebraic formalism, known as the Euclidean connection, to calculate the S matrix for systems whose Hamiltonian may be written as a function of the Casimir operator of an $so(2,1)$ algebra.

In this work we study the application of the Euclidean connection to obtain in a straightforward manner the S matrix for the HNP, by extending the treatment developed in [5].

We start with a brief review of [5] and then introduce the basic ideas of the Euclidean connection for $so(2,1)$ [1]. Then we study its application to the scattering sector of the HNP, obtaining the S matrix for this family of potentials.

2 Bound states

As is well known, the Hypergeometric Natanzon Potentials (HNP) [2] are given by:

$$\begin{aligned} V(z) &= \frac{f\,z^2-(h_0-h_1+f)\,z+h_0+1}{R(z)} \\ &\quad+\left(a+\frac{a+(c_1-c_0)\,(2z-1)}{z\,(z-1)}-\frac{5}{4}\frac{\Delta}{R(z)}\right)\frac{z^2\,(1-z)^2}{R(z)^2} \end{aligned} \tag{1}$$

where

$$R(z)=a\,z^2+\tau\,z+c_0 \quad \Delta=\tau^2-4\,a\,c_0 \quad \tau=c_1-c_0-a \tag{2}$$

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where f, h_0, h_1, a, c_0, c_1 are the *Natanzon Parameters* and $z(r)$ is a real arbitrary function that satisfies

$$\frac{dz}{dr} \equiv z' = \frac{2\,z\,(z-1)}{\sqrt{R(z)}} \qquad z(r) \in [0,1] \tag{3}$$

and the transformation $r(z)$ is assumed to take $r \to \infty$ to $z = 1$. In [5] the bound state sector of the HNP was studied by means of an $so(2,1)$ algebra, obtaining an algebraic description of it. The realization used is

$$\begin{aligned}
J_0 &= -i\partial_\phi \\
J_\pm &= e^{\pm i\phi}\left\{\pm\frac{\sqrt{z}(z-1)}{z'}\partial_r - \frac{1}{2}\frac{i(z+1)}{\sqrt{z}}\partial_\phi\right\} \\
&\quad \pm e^{\pm i\phi}\left\{\frac{1}{2}(z-1)\left[\frac{1\mp p_\nu}{\sqrt{z}} - \frac{z"\sqrt{z}}{z'^2}\right]\right\} \\
Q &= \frac{z\,(z-1)^2}{z'^2}\partial_r^2 + \frac{1}{4}\frac{(z-1)^2}{4\,z}\partial_\phi^2 + \frac{1}{2}\frac{i\,p_\nu\,(z^2-1)}{z}\partial_\phi \\
&\quad + \frac{(z-1)^2}{4}\frac{\left[z^2\,(2z'''z' - 3z")-z'^4\,(p_\nu^2-1)\right]}{z\,z'^4}
\end{aligned} \tag{4}$$

where p is a parameter, which allows the use of the $so(2,1)$ algebra to completely describe the HNP. The algebraic treatment makes use of

$$(Q-q)\Psi(r,\phi) = \mathcal{G}(r)(E-H)\Psi(r,\phi) \tag{5}$$

where H is the standard radial Hamiltonian, q is the eigenvalue of the Casimir operator Q and $\mathcal{G}(r)$ is a function of r. $\Psi(r,\phi)$ is simultaneously an eigenfunction of Q and J_0 as well as of H, and is given by $\Psi(r,\phi) = e^{im\phi}\Phi(r)$. For the bound states the $\mathcal{D}^{(+)}$ representation is used, therefore the compact generator J_0 has eigenvalues [6]

$$m = \nu + \frac{1}{2} + \sqrt{q_\nu + \frac{1}{4}} \qquad \nu = 0, 1, 2, ..., \nu_{max} \tag{6}$$

The energy spectrum is obtained from [5]:

$$2\nu + 1 \quad = \quad \alpha_\nu - \beta_\nu - \delta_\nu \tag{7}$$

where

$$\begin{aligned}
\alpha_\nu &\equiv \sqrt{-aE_\nu + f + 1} = p_\nu + m_\nu \\
\beta_\nu &\equiv \sqrt{-c_0 E_\nu + h_0 + 1} = p_\nu - m_\nu \\
\delta_\nu &\equiv \sqrt{-c_1 E_\nu + h_1 + 1} = \sqrt{4q_\nu + 1}
\end{aligned}$$

This are the basic equations of the algebrization derived in [5].

3 Euclidean connection

The main motivation for Euclidean connection [1] is that the asymptotic states (regular solution) are written as [7]:

$$|\phi\rangle^{(\infty)} = \mathcal{A}e^{ikr} + \mathcal{B}e^{-ikr} \tag{8}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the Jost functions and $e^{\pm ikr}$ are eigenfunctions of an Euclidean algebra $e(2)$:

$$[L_z, P_x] = iP_y \quad [L_z, P_y] = -iP_x \quad [P_x, P_y] = 0 \tag{9}$$

A realization in polar coordinates of the ladder and Casimir operators is as follows:

$$\begin{aligned} P_\pm &= e^{\pm i\phi}\left[-i\partial_r + \frac{1}{r}\left(\pm i\partial_\phi + \frac{1}{2}\right)\right] \\ \mathbf{P}^2 &= -\partial_r^2 - \frac{1}{r^2}\left(\partial_\phi^2 + \frac{1}{4}\right) \end{aligned} \tag{10}$$

where $P_\pm \equiv P_x \pm iP_y$ and $\mathbf{P}^2 \equiv P_+P_-$. Taking the limit $r \to \infty$, the asymptotic generators are obtained:

$$\begin{aligned} P_\pm^{(\infty)} = -ie^{\pm i\phi}\partial_r \qquad L_z^{(\infty)} = -i\partial_\phi \\ \mathbf{P}^{2\,(\infty)} = -\partial_r^2 \end{aligned} \tag{11}$$

that also closes an $e(2)$ algebra. The eigenstate basis $\left\{|\pm k, m\rangle^{(\infty)}\right\}$ is

$$|\pm k, m\rangle^{(\infty)} = e^{\pm ikr}e^{im\phi} \tag{12}$$

and the action of the asymptotic generators upon it is given by:

$$\begin{aligned} P_\pm^{(\infty)}|(\pm)k, m\rangle^{(\infty)} &= (\pm)k|(\pm)k, m \pm 1\rangle^{(\infty)} \\ \mathbf{P}^2|\pm k, m\rangle^{(\infty)} &= k^2\,|\pm k, m\rangle^{(\infty)} \end{aligned} \tag{13}$$

As is well known, we can expand the ladder operators of $so(2,1)$ in terms of the $e(2)$ generators [8]. Therefore, given an $so(2,1)$ algebrization of the bound states, it is natural to study its extension to the scattering sector by means of the mentioned expansion. The expanded ladder operator $J_+^{(\infty)}$ of $so(2,1)^{(\infty)}$ in the continuous representation (where $j = -\frac{1}{2} + i\,f(k)$) obtained in [1] is:

$$J_+^{(\infty)}(\pm k) = \frac{e^{i\gamma_\pm(k)}}{\pm k}\left[\left(-\frac{1}{2} \mp i f(k)\right)P_+^{(\infty)} + L_z^{(\infty)}P_+^{(\infty)}\right] \tag{14}$$

where $f(k)$ is defined by:

$$k^2 = h\left(-f^2(k)\right) \qquad E_j = h\left[q + \frac{1}{4}\right] \tag{15}$$

where $h(\eta)$ is the function that connects the Energy E_j with the Casimir eigenvalue q or, equivalently, the Hamiltonian with the Casimir operator Q.

A recurrence relation may be calculated and solved for the coefficients A_m and B_m by applying (14) directly to (8), with $|\phi\rangle^{(\infty)} = e^{-im\phi}|j,m\rangle^{(\infty)}$, and equating it to the action of $J_+^{(\infty)}$ upon $|j,m\rangle^{(\infty)}$. Since $S_m(k) = A_m/B_m$, then:

$$S_m(k) = e^{im[\gamma_+ + \gamma_-]} \frac{\Gamma\left[m+\frac{1}{2}-if(k)\right]}{\Gamma\left[m+\frac{1}{2}+if(k)\right]} \Delta(k) \tag{16}$$

where $\Delta(k)$ is a constant factor to be determined.

4 S matrix for the HNP

Starting from (4), the asymptotic algebra is readily obtained by taking the limit $r \to \infty$ (or equivalently $z = 1$).

$$\begin{aligned} J_0^{(\infty)} &= -i\,\partial_\phi \\ J_\pm^{(\infty)} &= e^{\pm i\phi}\left[\mp\frac{\sqrt{c_1}}{2}\,\partial_r - i\,\partial_\phi \pm \frac{1}{2}\right] \\ Q^{(\infty)} &= \frac{c_1}{4}\partial_r^2 - \frac{1}{4} \end{aligned} \tag{17}$$

Since these generators form also an $so(2,1)$ algebra, the Euclidean connection may be applied directly. Equation (17) corresponds to the $so(2,1)^{(\infty)}$ algebra mentioned before. We only need to determine the expansion coefficients $f(k)$ and $\gamma_\pm(k)$. This is accomplished by equating the action of (14) with the one of (17), obtaining

$$\begin{aligned} \gamma_\pm(k) = 0 \qquad & f(k) = \frac{k\sqrt{c_1}}{2} \\ \Rightarrow\quad & h'(\eta) = -\frac{4}{c_1}\,\eta \end{aligned} \tag{18}$$

From (7) we know that

$$\begin{aligned} E_j = -\frac{4\,q - h_1}{c_1} \quad &= \quad h\left[q + \frac{1}{4}\right] \\ \Rightarrow \quad & h(\eta) \equiv -\frac{4}{c_1}\,\eta + \frac{h_1+1}{c_1} \end{aligned} \tag{19}$$

Since we are dealing with the scattering sector and $V_{PNH} \propto h_1 + 1$ as $r \to \infty$ [2], $h_1 = -1$. Therefore $h'(\eta) = h(\eta)$ and the S matrix is given by (16):

$$S_m(k) = \frac{\Gamma\left(m+\frac{1}{2}-\frac{1}{2}\,ik\sqrt{c_1}\right)}{\Gamma\left(m+\frac{1}{2}+\frac{1}{2}\,ik\sqrt{c_1}\right)} \Delta_m(k) \tag{20}$$

where $\Delta_m(k)$ is a regular function of k derived by Natanzon [2]. It can easily be verified that the poles are in agreement with the bound state energy spectrum.

5 Conclusions

We have successfully extended the algebraic treatment for the bound states developed in [5] to the scattering sector via the $so(2,1)$ Euclidean connection, therefore obtaining a purely algebraic description of the HNP using an $so(2,1)$ algebra. This can also be accomplished with a methodology analogous to the one used by Frank, et al. [9]. This was done in detail and it is to be published soon [10]. Meanwhile, a short review may be found in this volume [11]. Even though Alhassid, et al. had obtained an algebraic description of the HNP in [12] by means of an $so(2,2)$ algebra, it is important to notice that our treatment is simpler.

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