## Option Pricing Using Arbitrage and Stochastic Calculus

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# 1 Introduction

Undergraduate mathematics majors are familiar with the differential and integral calculus first developed by Newton and Leibniz during the latter 17th century. This calculus is used to analyze deterministic (rather than random) functions $f(x)$. Development of this calculus (which we will henceforth refer to as "deterministic calculus") and its applications has indeed been one of the most fruitful ideas in scholarly pursuits as evidenced by the depth and breadth of differential equation models. When it comes to modern mathematical finance in which price uncertainties are of central importance, *stochastic calculus*, the calculus of random variables, becomes foundational. We will see, however, that the Taylor series from deterministic calculus provides a good starting point for understanding the modifications that are needed in analyzing instantaneous changes in random variables.

In this Module we focus on a single, important problem in mathematical finance, the valuation or pricing of options. We begin in Section 2 by explaining the concept of *arbitrage*. This section, which can be understood without prior knowledge of math finance, considers only problems which do not involve stochastic calculus, and provides an important heuristic for pricing options. In Section 3, we introduce elementary tools from stochastic calculus that serve as a starting point for describing changes in stock and option prices. Although we have assumed basic knowledge of probability such as expected values and variance, we introduce further concepts (which might supplement a first semester course) including *stochastic processes*, *mean square limits* and *Ito integrals* which are necessary to understand the *stochastic differential equations* (SDE's) used to describe instantaneous changes involving random variables. In Section 4, we see how the concepts of arbitrage and stochastic differential equations come together in the foundational Black-Scholes equation for option valuation. Section 5 points the reader to several reference books which provide more complete information and background for the topics introduced in this Module. The exercises form an integral part of this module, and most of their solutions have been included.

# 2 Arbitrage Valuation of Options

In this section, we introduce several basic finance concepts such as *present value*, *selling a stock short*, and *options*, and then give several examples (which do not require the use of stochastic calculus) how the value of an option may be determined by a concept called *arbitrage*. We refer the reader to Ross[2003] for a more complete treatment of this material.

## 2.1 Present Value

If an amount $x$ is put into an account with simple interest rate $r$ per period, then after a single compounding of interest, the original amount increases to $x(1+r)$. For an amount $A$ to grow to some specified amount $A_1$ after one period, a certain interest rate $r^*$ produces such growth. This rate $r^*$ is called the *rate of return* (see **Exercise 2.1.2**).

More generally, if an amount $x$ is put into an account with simple interest rate $r$ per period, then after $n$ compoundings of interest, this original amount increases to $x(1+r)^n$. It follows that in order for $x$ to grow to an amount $y$ after $n$ periods, we must have

$$\begin{aligned} x(1+r)^n &= y \Rightarrow \\ x &= y(1+r)^{-n}. \end{aligned}$$

The amount $x = y(1+r)^{-n}$ is called the *present value* of the amount $y$ which is $n-$ periods in the future.

The concept of present value also applies to a situation where we create a (fixed) *income stream* in which we will withdraw a fixed amount $A$ at the end of each period for $n$ periods. The present value of this income stream is the amount $x$ which we must deposit into the account at the present time in order to cover these future withdrawals. In this case, the amount $x$ satisfies

$$x = A\beta + A\beta^2 + \cdots + A\beta^n = A\beta\frac{1-\beta^n}{1-\beta},$$

where $\beta = \frac{1}{1+r}$.

**Exercises**

**2.1.1** Suppose we wish to withdraw an amount $A_i$ at the end of each period $i$ ($i = 1, \cdots, n$) from an account with interest rate $r$ per period. Find the present value of the (variable) income stream $A_1, A_2, \cdots, A_n$, that is, the amount $x$ which must be on deposit at the beginning of the first period to cover these future withdrawals.

**2.1.2** **a)** Suppose an amount $A$ grows to an amount $A_1$ after one compounding of interest. Find the *rate of return*, that is, the interest rate $r^*$ which produces this growth.

**b)** Find a quadratic equation satisfied by $\beta = \frac{1}{1+r^*}$ where $r^*$ is the rate of return for an initial amount $x$ which generates the two-period (variable) income stream $A_1, A_2$.

**2.1.3** Suppose interest is compounded at a nominal annual rate $r$ so that the rate per period is $r/n$ where $n$ is the number of periods per year. Continuous compounding of interest is obtained by the limit $n \to \infty$.

**a)** Show that under continuous compounding of interest, an initial amount $A_0$ will grow to an amount $A_0 e^{rt}$ at time $t$.

**b)** Assuming continuous compounding of interest at nominal rate $r$, what is the present value $x$ of an amount $y$ at a future time $t$ units from the present?

## 2.2 Arbitrage Pricing of a Call Option

An *option* is a type of *derivative security*, meaning, a financial contract whose value depends on the value of an underlying *cash market asset* such as a stock or bond. Depending on the type of option purchased, one has the right (but is not obligated) to buy or sell the underlying asset at a specified price at or before a given time of expiry:

- *European Call Option*: Gives the right at time of expiry to *buy* the cash market asset at a price $K$ (called the *strike price*);

- *European Put Option*: Gives the right at time of expiry to *sell* the cash market asset at price $K$ (called the *exercise price*);

- *American Call Option*: Gives the right at any time up through the time of expiry to buy the cash market asset at strike price $K$; and

- *American Put Option*: Gives the right at any time up through the time of expiry to sell the cash market asset at exercise price $K$.

An *arbitrage* is an unfair advantage in an investment situation. Some basic types of arbitrage include

- an investment opportunity that guarantees *without risk* a rate of return (see *Exercise 2.1.2*) which is higher than a specified standard risk-free return rate such as a U.S. Treasury bill rate (as discussed in this section).

- A betting strategy in a game of chance which has a positive expected return (as explained in Section 2.4)

- a transaction that involves no negative return at any probabilistic market state and a positive return in at least one state (see [Etheridge 2002] for a discussion of this type of arbitrage).

One way to show that an arbitrage exists is to demonstrate an investment scheme that is guaranteed to make money without needing any money to begin with (all money needed for the investment is borrowed initially and is repaid at the end using the risk-free interest rate). A central question in the valuation of options is the determination of prices which preclude arbitrage opportunities.

Let $t = 0$ denote the present time, and $t = 1$ the next time at which a transaction involving a certain stock can be made. Furthermore, let us assume that the price of the stock at $t = 0$ is $P_0$, and, for simplicity, at $t = 1$ the new price $P_1$ may be one of only two possibilities:

- the new price $P_1$ has risen to $P_1 = P_{up}$; or,

- the new price $P_1$ has fallen to $P_1 = P_{down}$.

Initially, we assume that we do not own any stock. In what follows, we will restrict our attention to the following transactions:

**Buying or selling the stock:** Let $x$ denote the number of shares of stock that we purchase ($x > 0$) or sell ($x < 0$) at $t = 0$. The term *selling the stock short* is used to describe sale of a stock which we do not own. If we sell the $x$ shares of stock short to another party, then at $t = 0$ we initially obtain from that party an amount $-xP_0$ with the agreement that we will "buy back" the stock by paying that party an amount $-xP_1$ at time $t = 1$.

**Buying or selling a call option:** For a cost $C$ per share, we can purchase at time $t = 0$ an option to buy $y$ shares at time $t = 1$ at a unit price $P_{call}$. To purchase this option we must pay $Cy$ at $t = 0$. If $P_1 > P_{call}$, we can make use of our option and *call* for the stock at *strike price* $P_{call}$ and thereby generate revenue in the amount $(P_1 - P_{call})y$. The option is worthless if $P_1 \leq P_{call}$.

As we mentioned earlier, an arbitrage exists if there is a risk-free way to obtain a higher rate of return than a specified "risk-less" rate. We will now use arbitrage considerations to obtain the price of a European call option:

**Lemma 1** Let $r$ be the standard risk-less interest rate for the option period and for simplicity assume further that $P_{down} < P_{call} < P_{up}$. In order to preclude arbitrage, there is only one possible unit price $C$ for the European call option, namely

$$C = \frac{P_{up} - P_{call}}{P_{up} - P_{down}}[P_0 - \frac{P_{down}}{1+r}].$$

**Proof:** We will show that unless $C$ is as stated in the lemma, there is an investment strategy that creates an arbitrage. To this end, let $x$ be the units of stock and $y$ be the units of options purchased at $t = 0$. The cost of this initial transaction is $A_0 = P_0x + Cy$. If $A_0 > 0$, this money is borrowed at interest rate $r$ so the amount which must be repaid at $t = 1$ is $A_1 = A_0(1+r)$. If $A_0 < 0$, this amount is used as a risk-free investment and so grows to $-A_1 = -A_0(1+r)$ at time $t = 1$.

At $t = 1$, the value $V$ of the $x$ units of stock and $y$ units of options depends on whether the stock price $P_1$ has gone up or down:

- **If** $P_1 = P_{up}$**:** $V = P_{up}x + (P_{up} - P_{call})y$.
- **If** $P_1 = P_{down}$**:** $V = P_{down}x$.

Note that we can choose $y$ so that $V$ does not depend on the value of $P_1$; in other words, choose $V$ so that

$$\begin{aligned} P_{up}x + (P_{up} - P_{call})y &= P_{down}x \Rightarrow \\ y &= -x\frac{P_{up} - P_{down}}{P_{up} - P_{call}}. \end{aligned}$$

For this choice of $y$, our net gain at $t = 1$ will be

$$\begin{aligned} gain &= V - A_1 \\ &= P_{down}x - (P_0x + Cy)(1+r) \\ &= (1+r)x[\frac{P_{down}}{1+r} - P_0 + C\frac{P_{up} - P_{down}}{P_{up} - P_{call}}]. \end{aligned}$$

We conclude that if $C$ is anything other than what is specified by the lemma, we can compute at $t = 0$ the non-zero value of the quantity $\frac{P_{down}}{1+r} - P_0 + C\frac{P_{up}-P_{down}}{P_{up}-P_{call}}$ and then choose $x$ to be either positive or negative so as to ensure that this gain at $t_1$ is positive without needing any money of our own at $t = 0$.

A second method for obtaining the option price $C$ is called *replicating the claim.* By this we mean the following. Suppose we sell an option for price $C$ and use this money to construct a portfolio consisting of $x_1$ dollars and $x_2$ units of stock. (This means we valued the option as $V(x_1, x_2) = x_1 + x_2 P_0$).) We would like the value of our portfolio to cover a possible claim by the optional holder at the end of the first period. (This is referred to as *hedging the contingent claim*). If the stock price has gone up, we need $P_{up} - P_{call}$ dollars to cover the claim, which implies

$$x_1(1+r) + x_2 P_{up} = P_{up} - P_{call}.$$

On the other hand, if the stock price has gone down, the option is worthless, which implies

$$x_1(1+r) + x_2 P_{down} = 0.$$

Solving this system simultaneously yields the solution

$$x^* = \frac{-P_{down}(P_{up} - P_{call})}{(P_{up} - P_{down})(1+r)}, \qquad y^* = \frac{P_{up} - P_{call}}{P_{up} - P_{down}}.$$

It is then straightforward to check that

$$V(x_1^*, x_2^*) = \frac{P_{up} - P_{call}}{P_{up} - P_{down}}[P_0 - \frac{P_{down}}{1+r}].$$

**Exercises**

**2.2.1** Suppose the unit value of a stock is 100 at $t = 0$ and either 200 or 50 at $t = 1$. Use arbitrage pricing to find the unit price of an option that calls for the unit price of stock to be 150 at time $t = 1$. Assume the risk-free interest rate for one period is 1/9.

**2.2.2** Suppose the current price of a stock is 100 and the payoff at time $t = 1$ of a call option is 50 if the stock price is 200 and 0 if the price is 50. We claim that unless the unit cost $C$ of this call option satisfies $C = C^* = (100 - \frac{50}{1+r})/3$, an arbitrage exists.

**a)** If $C < C^*$, show that an arbitrage opportunity is created by initially selling short 1/3 share, investing $C$ in the call option, and putting the remainder $A_0$ in the bank.

**b)** Describe an arbitrage opportunity for the case where $C > C^*$.

**2.2.3*** Suppose you own an option to buy one share of the stock at price $K$, and the option expires at time $t = T_{exp}$. An American style call allows you to make use of the option at any time up through expiration (i.e. $0 < t \leq T_{exp}$), whereas a European style call requires you to wait until the time of expiration before using the call. Is the American style call worth more than the European style?

**2.2.4** Suppose the risk free rate per period is $r$ and a stock whose price is currently $P_0$ will either rise to $P_1 = P_0u$ or $P_1 = P_0d$ (where $d < 1+r < u$. Let $(P_1 - P_{claim})_+$ denote the amount needed to cover the claim by a holder of an option with strike price $P_{claim}$. (The notation $Q_+$ is equal to $Q$ if $Q > 0$ and 0 otherwise.) Use the method of replicating the claim to show that the no-arbitrage option price is

$$\frac{1 - \frac{d}{1+r}}{u-d}(P_0u - P_{claim})_+ + \frac{\frac{u}{1+r} - 1}{u-d}(P_0d - P_{claim})_+.$$

## 2.3 Put-Call Parity

Let $t = 0$ be the current time, and $S$ the current price of a stock. We have just examined a (European) call option with unit price $C$, one which gives the owner the right to *purchase* the stock at price $K$ at time of expiry $T_{exp}$. Let $P$ be the unit price of a (European) *put* option, that is, one which gives the owner the right to *sell* the stock at exercise price $K$ at time of expiry $T_{exp}$. As one might suspect, in order to preclude arbitrage, there is a constraining relationship between the respective call and put option prices $C$ and $P$:

**Put-Call Parity Relation** Assuming that interest is continuously discounted at a nominal rate $r$ (see **Exercise 2.1.3**), to avoid arbitrage, we must have

$$S + P - C = Ke^{-rT_{exp}}.$$

**Proof.** Let $\Delta = S + P - C$. The result follows from the claim that an arbitrage can be created by the following investment strategies:

- If $\Delta > Ke^{-rT_{exp}}$: Sell one share of stock, sell one put option, and buy one call option.
- If $\Delta < Ke^{-rT_{exp}}$: At $t = 0$, borrow an amount $\Delta$ from the bank, buy one share of stock, buy one put option, and sell one call option; and

We proceed to explain the first arbitrage opportunity and then leave the second as **Exercise 2.3**. At $t = 0$, by selling one share of stock, selling one put option (with exercise price $K$) and buying one call option (with strike price $K$), we realize a positive amount $\Delta = S + P - C$ which we deposit in the bank. This initial deposit grows to $(S + P - C)e^{rT_{exp}} > K$ at time $T_{exp}$. Let $S_{exp}$ be the price of the stock at time $T_{exp}$. If $S_{exp} < K$, our call option is worthless and the put option we sold will be exercised forcing us to spend an amount $K$ by which we acquire the one share of stock we initially sold short. This leaves us with a positive gain: $(S+P-C)e^{rT_{exp}} - K > 0$. If $S_{exp} \geq K$, then the put option we sold is worthless and we can use our call option to purchase the one share of stock at unit price $K$ in which case we are again left with a positive gain: $(S+P-C)e^{rT_{exp}} - K > 0$.

**Exercise**

**2.3** Prove an arbitrage exists when $\Delta = S + P - C < Ke^{-rT_{exp}}$.

## 2.4 A Binomial Model

Up to this point, we have restricted our attention to change in stock price over a single period $0 \leq t \leq T_{exp}$. Let us now consider what happens if we track the price of a stock over an $n$ period time horizon, where for simplicity we asssume a fixed nominal interest rate $r$ over this entire time span. The stock may be purchased or sold at the end of any or all of the $n$ periods.

Let $S_i$ denote the price of the stock at the end of period $i$ so that $S_0$ and $S_n$ represent respectively the intial and final stock price over the entire time horizon. In constructing what is called the *binomial model*, we make a further simplifying assumption that over each period, the price of the stock either goes up by a factor $u$ (with $u > 1 + r$) or down by a factor $d$ (with $0 < d < 1$). In other words, for each $i = 1, 2, \cdots, n$, either

- $S_i = uS_{i-1}$; or
- $S_i = dS_{i-1}$.

Digressing for a moment, let us describe an arbitrage opportunity by considering a fictitious casino game in which players bet on which card will be randomly drawn from a deck of 100 cards. In a fair deck, all the cards are distinct and in a fair game (players have zero expected return) the casino will pay \$100.00 for each \$1.00 winning bet. An arbitrage (betting strategy with positive expected return) would be created, if, for example, the deck were stacked with two of the same card, and a person with inside information always bet on that card.

Returning to the multi-period binomial model, let us define a vector $< X_1, X_2, \cdots, X_n >$ where $X_i = 1$ (respectively $X_i = 0$) if the stock goes up in price (respectively down in price) between the $i - 1^{st}$ and $i^{th}$ period. We will now argue that in order to avoid an arbitrage opportunity (investment strategy with positive expected return), in the multi-period binomial model, for each $i = 1, \cdots, n$, the probability that $X_i = 1$ must be given by

$$p = \frac{1 + r - d}{u - d}.$$

To this end, fix an arbitrary value for $i$. Consider an investment strategy which randomly selects a sequence $x_1, \cdots, x_{i-1}$ of $0's$ and $1's$, and then says that if $X_j = x_j$ for all $j = 1, \cdots, i-1$, then buy one unit of stock at the end of the $i-1^{st}$ period and sell it back at the end of the next (that is, the $i^{th}$) period. Let $\alpha$ be the probability that the stock is purchased at the end of perod $i-1$; that is, $\alpha = P(X_1 = x_1, \cdots, X_{i-1} = x_{i-1})$. The probability that the stock we so purchased will then go up in price is given by the conditional probability

$$\beta = P(X_i = 1 | X_1 = x_1, \cdots, X_{i-1} = x_{i-1}).$$

The present value of the expected gain (taking "present" to mean the time of investment, i.e., the end of the $i - 1^{st}$ period) is given by

$$\alpha[\beta(1+r)^{-1}uS_{i-1} + (1-\beta)(1+r)^{-1}dS_{i-1} - S_{i-1}].$$

To avoid arbitrage, this expected gain must be zero, so that

$$\frac{\beta u}{1+r} + \frac{(1-\beta)d}{1+r} = 1 \Rightarrow \beta = \frac{1+r-d}{u-d}.$$

Since the $x_i$ are arbitrary, to avoid arbitrage, it is necesary that $p = \beta = \frac{1+r-d}{u-d}$. (Beyond the scope of this module, one can actually show that this value of $p$ is also sufficient to preclude arbitrage on any investment scheme. That is, all such schemes have zero expected gain.)

## Exercise

**2.4** For the multi-period binomial model described in this section, consider now the problem of valuation of a call option obtained for the unit price $C$ at the beginning of the first period which allows the owner the right to purchase the stock for unit price $K$ at the end of the $n^{th}$ period.

- **a)** Assuming that the $X_i$ are independent Bernoulli random variables with probability $p$, it follows that $Y = \sum_{i=1}^{n} X_i$ is a binomial random variable with parameters $n$ and $p$. Show that $S_n = u^Y d^{n-Y} S_0$.
- **b)** Find the value $V$ of the option at the end of the $n^{th}$ period.
- **c)** Find in terms of $E(V)$ the expected present value of owning the option, and hence the arbitrage valuation of $C$. (Take "present" to mean the beginning of the first period.)

# 3 Stochastic Calculus and Instantaneous Price Changes

In this section we present a brief overview of basic concepts from stochastic calculus and how this calculus is applied to describing instantaneous changes in the price of a stock or option. Along with the previous section which developed the idea of arbitrage, the material in this section will provide necessary background for our introduction in Section 4 of the celebrated Black-Scholes equation. We refer the reader to Neftci [1996] for a more complete presentation of the stochastic calculus.

## 3.1 Stochastic Processes

The type of function which is analyzed in stochastic calculus is called a *stochastic process*, and has the form

$$t \to x_t$$

where for each value of the real *index value* $t$ (which for our purposes will always denote time), $x_t$ is a random variable.

Stochastic processes may be used in composition to form new stochastic processes. In the sequel we will model option prices using stochastic processes of the form

$$t \to F(x_t, t),$$

where $x_t$ is the price of a stock, and $F(x_t, t)$ the price of a related option, at time $t$. Note that while the stock price $x_t$ is a random variable, once the value of $x_t$ is revealed, the corresponding option price is determined by the function $F(x_t, t)$. For each possible stock price trajectory given by revealed values of $x_t$, there corresponds a single option price trajectory.

Though the random nature of the stochastic process $x_t$ describing stock price means there are many different possible trajectories in the stock price, and consequently also in the option price, a specific option price trajectory based on a specific stock price trajectory may be differentiable in the deterministic calculus sense. **Exercise 3.1** utilizes deterministic calculus to analyze a *risk-free* (sometimes called *risk-neutral*) stock/option investment strategy known as *Delta hedging*. This strategy offsets a potential loss in selling stock short by taking the opposite position in call options.

**Exercise**

**3.1** Suppose a trajectory $g(x) = F(x_t, t)$ for a call option written on a stock with revealed price trajectory is a differentiable function. Show that when the stock price is $x_0$, purchasing one unit of a call option will offset a loss incurred by selling short $g'(x_0)$ units of stock.

## 3.2 Risk

Though we used deterministic calculus to analyze Delta hedging, in general, this calculus is limited in its direct applicability to option pricing. The main reason is the existence of *risk*.

That stock price $x_t$ is a stochastic process implies uncertainty as to whether the stock price will move up or down and by how much. Investment strategies must take into account the risk that a stock price will drop. Mathematically, risk in stock price is described using a random variable with a positive variance: the greater the variance, the more risky the investment since there is more chance that a greater size loss will occur over a given time period.

If deterministic calculus were sufficient to analyze all possible option prices $g(x)$ considered to be smooth functions of the underlying stochastic stock price $x = x_t$, then using the second order Taylor approximation about stock price $x = x_0$, we would have

$$\Delta g = g(x_0 + \Delta x) - g(x_0) \approx g'(x_0)\Delta x + \frac{1}{2}g''(x_0)(\Delta x)^2, \tag{1}$$

where $\Delta x = x - x_0$. As $\Delta x \to 0$, the term $\frac{1}{2}g''(x_0)(\Delta x)^2 \to 0$, which would imply that $E[(\Delta x)^2] \to 0$.

On the other hand, $VAR[x] = VAR[\Delta x] = E[(\Delta x)^2] - (E[\Delta x])^2$. As $\Delta x \to 0$, $E[\Delta x] \to 0$, so $VAR[x] \to E[(\Delta x)^2]$. Thus, $E[(\Delta x)^2]$ must remain positive as $\Delta x \to 0$ in order for the random variable $x$ to model risk. The Taylor approximation (1) reveals the need for a new type of calculus in analyzing option prices.

## 3.3 Continuous Time Martingales and Wiener Processes

Consider a stochastic process $x_t$ for time $t$ belonging to a finite interval $[0, T]$. Let $I_s$ denote an *information set*, that is, all the relevant information available up to time $s \in [0, T]$ for determining the value of $x_t$. In what follows, we will use the following terminology and notation:

- $x_t$ is said to be *$I_t$-adapted* if the value of $x_t$ is included in $I_t$;
- The expected value of $x_t$ given the information set $I_s$ is denoted $E_s[x_t] = E[x_t|I_s]$; and
- $E[x_t] = E_0[x_t]$ is called the *unconditional* expected value of $x_t$.

Note that the above imply that if $x_t$ is $I_t$ adapted, then $E_t[x_t] = x_t$, since the value of $x_t$ is known.

Given a probability distibution $P$ used to compute expected values, we make the following definition:

DEFINITION: A stochastic process $x_t$ is called a *continuous martingale* with respect to a family of information sets $I_t$ and probability distribution $P$ if for all $t \in [0, T]$

- $x_t$ is $I_t$ adapted;
- $E[x_t] < \infty$ (all unconditional expectations are finite); and
- $E_t[x_T] = x_t$ (the best forecast of $x_T$ is the last observed value $x_t$).

If, in addition, $E[x_t^2] < \infty$ for all $t$, the continuous martingale $x_t$ is said to be *square integrable*.

**Exercise**

**3.3.1 a)** Show that if $x_t$ is a martingale and $0 \le s < t \le T$, then $E_s[x_t - x_s] = 0$.

**b)** Explain why if $T$ is large, then $x_t$, the long range price of a stock for $t \in [0, T]$, is not ordinarily a martingale process.

Now suppose that $x_t$ is a martingale defined for $t \in [0, T]$, let $n$ be a positive integer and $t_i = \frac{T}{n} i$. Define the *martingale difference* $\Delta_{i+1} x \equiv x_{t_{i+1}} - x_{t_i}$. Using **Exercise 3.3a)**, we see that $E_{t_i}[\Delta_{i+1} x] = 0$. That is, using the currently available information, the expected change in value of a martingale as time progresses is 0.

In the sequel, we will investigate a special type of stochastic process $x_t$ called a *Wiener Process*; that is, a continuous square integrable martingale $x_t$ with respect to information sets $I_t$ and probability distribution $P$ such that:

- $x_0 = 0$;
- For each $s \ge 0$ and $t > 0$, the random variable $x_{t+s} - x_s$ is normally distributed with mean zero and variance $t$;
- For each $n \ge 1$ and any times $0 \le t_0 \le t_1 \le \cdots \le t_n$, the random variables $x_{t_r} - x_{t_{r-1}}$ are independent;
- The random variable $x_t$ changes continuously with $t$.

Properties of Wiener processes include [Neftci 1996]:

- $E[x_t] = 0$;
- $E[x_t^2] = t$;
- $E[(x_t - x_s)^2] = t - s$ whenever $s < t$; and
- $E[(\Delta_{i+1}x)^2(\Delta_{j+1}x)^2] = (t_{i+1} - t_i)(t_{j+1} - t_j)$ (for $i \neq j$); and
- $E[(\Delta_{i+1}x)^4] = 3(t_{i+1} - t_i)^2$.

**Exercise**

**3.3.2** Prove the following properties stated above for a Wiener process $x_t$:

**a)** $E[(x_t - x_s)^2] = t - s$ whenever $s < t$.

**b)** $E[(\Delta_{i+1}x)^4] = 3(t_{i+1} - t_i)^2$.

## 3.4 Mean Square Limits

Let $V_n$ denote a sequence of random variables. A random variable $V$ is called the mean square limit of the sequence $V_n$ if

$$\lim_{n\to\infty} E[(V_n - V)^2] = 0.$$

Let $x_t$ denote a Wiener process defined on an interval $[0, T]$ and for a fixed positive integer $n$, define $t_i = \frac{T}{n}i$. Next, define $\Delta_{i+1}x = x_{t_{i+1}} - x_{t_i}$ . We claim that the mean square limit as $n \to \infty$ of the random variable $V_n = \sum_{i=0}^{n-1}(\Delta_{i+1}x)^2$ is equal to $T$. In other words, we have

**Lemma 2** Let $x_t$ be a Wiener process defined for $t \in [0, T]$. Then

$$\lim_{n\to\infty} E[(\sum_{i=0}^{n-1}(\Delta_{i+1}x)^2 - T)^2] = 0. \quad (2)$$

Proof: Let $V_n = \sum_{i=0}^{n-1}(\Delta_{i+1}x)^2$. The following computation shows that $E[V_n] = T$:

$$\begin{aligned} E[V_n] &= E[\sum_{i=0}^{n-1}(\Delta_{i+1}x)^2] \\ &= \sum_{i=0}^{n-1} E[(\Delta_{i+1}x)^2] \\ &= \sum_{i=0}^{n-1}(t_{i+1} - t_i) \\ &= T. \end{aligned}$$

The desired result follows immediately from the following:

**Exercise**

**3.4** Show that

$$E[(V_n - T)^2] = 2T^2/n.$$

## 3.5 The Ito Integral

Given a stochastic process $x_t$ defined for $t \in [0, T]$, the *Ito Integral*

$$\int_0^T f(x_t, t)dx_t$$

is defined to be the mean square limit as $n \to \infty$ of the sequence of random variables

$$V_n = \sum_{i=0}^{n-1}[f(x_{t_i}, t_i)\Delta_{i+1}x]$$

where $t_i = \frac{T}{n}i$ and $\Delta_{i+1}x = x_{t_{i+1}} - x_{t_i}$

In some cases, the Ito integral agrees with the familiar Riemann integral. For example, for any constant $k > 0$, $\int_0^T kdt = kT$. This holds since in this case $f(x_{t_i}, t_i) = k$ and $x_t = t$, so we have

$$V_n = \sum_{i=0}^{n-1}[k(t_{i+1} - t_i)] = kT.$$

However, in general, it is very difficult to determine or verify Ito integrals directly from the definition. (We will later see how Ito's Lemma (Section 3.8) provides an easier way to obtain certain Ito integrals.)

For example, let $x_t$ be a Wiener process. Let us use the definition of the Ito integral to verify that $\int_0^T x_t dx_t = \frac{1}{2}[x_T^2 - T]$.

To begin, note that

$$\begin{aligned} V_n &= \sum_{i=0}^{n-1}[x_{t_i}\Delta_{i+1}x] \\ &= \frac{1}{2}\sum_{i=0}^{n-1}[(x_{t_i}+\Delta_{i+1}x)^2 - x_{t_i}^2 - (\Delta_{i+1}x)^2] \\ &= \frac{1}{2}\sum_{i=0}^{n-1}[x_{t_{i+1}}^2 - x_{t_i}^2 - (\Delta_{i+1}x)^2] \\ &= \frac{1}{2}[x_T^2 - \sum_{i=0}^{n-1}(\Delta_{i+1}x)^2] \end{aligned}$$

Lemma 2 (Section 3.4) states that the mean square limit of $\sum_{i=0}^{n-1}(\Delta_{i+1}x)^2$ is equal to $T$. It follows that

$$\begin{aligned} E[(V_n - \frac{1}{2}[x_T^2 - T])^2] &= \frac{1}{4}E[(\sum_{i=0}^{n-1}(\Delta_{i+1}x)^2 - T)^2] \\ &= 0. \end{aligned}$$

It is important to note that Ito integrals may be expressed as a *stochastic differential equations* (SDE's). For example, if we set $F(x_t, t) = \frac{1}{2}x_t^2$, then (assuming $F(x_0, 0) = 0$) the Ito integral $\int_0^T x_t dx_t = \frac{1}{2}[x_T^2 - T]$ is equivalent to the SDE

$$x_t dx_t = dF(x_t, t) - \frac{1}{2}dt,$$

or, re-arranging terms,

$$dF(x_t, t) = x_t dx_t + \frac{1}{2}dt.$$

SDE's are defined by the corresponding Ito integrals, so that equality is understood to mean equality of the mean square limits.

**Exercise**

**3.5** Let $x_t$ be a Wiener process. Show that

**a)** $\int_0^T (dx_t)^2 = \int_0^T dt$.

**b)** $x_t$ satisfies the SDE $(dx_t)^2 = dt$.

## 3.6 Stochastic Differential Equations for Stock Price

The two types of SDE's we are most concerned about are

- SDE's which describe a stock price $x_t$; and
- SDE's which describe the related option price $F(x_t, t)$.

In this sub-section we describe the former, and then in the next, will see how *Ito's Lemma* is used to obtain the latter.

SDE's for stock price $x_t$ ($t \in [0, T]$) are of the form

$$dx_t = a(x_t, t)dt + \sigma(x_t, t)dW_t,$$

where $W_t$ is an $I_t$ adapted Wiener process. Intuitively, this SDE says that change in stock price is the sum of two components:

- $a(x_t, t)dt$, an expected upward "*drift*" in price; and
- $\sigma(x_t, t)dW_t$, an unpredictable change or "*volatility*".

The equivalent Ito integral relation is

$$\int_t^{t+h} dx_u = \int_t^{t+h} a(x_u, u)du + \int_t^{t+h} \sigma(x_u, u)dW_u.$$

## 3.7 Statement of Ito's Lemma

Ito's lemma is foundational in the understanding of stochastic differential equations and is useful in obtaining Ito integrals. The lemma assumes that stock price $x_t$ is given by an SDE of the form

$$dx_t = a(x_t, t)dt + \sigma(x_t, t)dW_t,$$

and then obtains an SDE satisifed by the option price $F(x_t, t)$. Before stating Ito's lemma, we follow Neftci [1996] by first giving the following intuitive line-of-reasoning:

- treating the stock price $x_t$ as if it were a deterministic variable (along with time $t$), we apply the deterministic two variable Taylor expansion to the option price $F(x_t, t)$; and then

- using a probabilistic criteria of what it means for a term in the expansion to be "negligible", we obtain an SDE relating the differentials $dF_t$, $dx_t$ and $dt$.

Partition a time interval $[0, T]$ into $n$ equal parts, let $\Delta t = T/n$ and $t_k = k\Delta t$ $(k = 0, \cdots, n)$. We will simplify our notation slightly by setting (for each $k$)

- $x_k = x_{t_k}$;
- $\mathbf{P_k} = (x_k, t_k)$;
- $\Delta_k x = x_k - x_{k-1}$; and
- $\Delta_k F = F(\mathbf{P_k}) - F(\mathbf{P_{k-1}})$.

The second order Taylor expansion for $F(x_t, t)$ about the point $\mathbf{P_{k-1}} = (x_{k-1}, t_{k-1})$ implies that

$$\begin{aligned}\Delta_k F \;\approx\; & \frac{\partial F}{\partial x}(\mathbf{P_{k-1}})\Delta_k x + \frac{\partial F}{\partial t}(\mathbf{P_{k-1}})\Delta t + \\ & \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[\Delta_k x]^2 + \frac{\partial^2 F}{\partial x \partial t}(\mathbf{P_{k-1}})\Delta_k x \Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\mathbf{P_{k-1}})[\Delta t]^2\end{aligned}$$

A term $\tau$ in this expansion is deemed negligible if the mean square limit as $\Delta t \to 0$ of the ratio

$$\frac{\tau}{\Delta t}$$

is equal to zero. For example, the term $\frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\mathbf{P_{k-1}})[\Delta t]^2$ is considered negligible since the ratio

$$\frac{\frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\mathbf{P_{k-1}})[\Delta t]^2}{\Delta t} = \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\mathbf{P_{k-1}})\Delta t$$

vanishes in the mean square limit $\Delta t \to 0$.

A key term which is non-negligible is

$$\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[\Delta_k x]^2.$$

It is instructive to examine why the ratio

$$\frac{\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[\Delta_k x]^2}{\Delta t}$$

does not vanish in the mean square limit $\Delta t \to 0$. Consistent with the previously mentioned hypothesis assumed by Ito's Lemma, we make the following assumption about the change in stock price $\Delta_k x$:

$$\Delta_k x = a_k \Delta t + \sigma_k \Delta_k W,$$

where (as described in the previous sub-section)

- $a_k \Delta t = a(x_{k-1}, t_k)\Delta t$ is an expected "upward drift" in price during the time interval $[t_{k-1}, t_k]$; and
- $\sigma_k \Delta_k W = \sigma_k (W_k - W_{k-1})$ is a random price "volatility" which accounts for unpredictable change in price during the same time interval. (We will refer to $\Delta_k W$ as the *innovation term*).

The innovation term has been carefully studied (see Neftci [1996]). Here we simply mention two facts:

- $E[\sigma_k \Delta_k W] = 0$; and
- $VAR[\sigma_k \Delta_k W] = E[(\sigma_k \Delta_k W)^2] = \sigma_k^2 \Delta t$.

These facts imply that for small $\Delta t$,

$$\begin{aligned}
\frac{\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[\Delta_k x]^2}{\Delta t} &= \frac{1}{2}\frac{\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[a_k \Delta t + \sigma_k \Delta_k W]^2}{\Delta t} \\
&= \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[a_k^2 \Delta t + 2a_k \sigma_k \Delta_k W + \frac{(\sigma_k \Delta_k W)^2}{\Delta t}] \\
&\approx \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})\sigma_k^2.
\end{aligned}$$

The last approximation indicates why the term $\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[\Delta_k x]^2$ is not negligible as $\Delta t \to 0$.

In **Exercise 3.7**, you are asked to show that after discarding all negligible terms in the second order Taylor expansion, we have

$$\Delta_k F \approx \frac{\partial F}{\partial x}(\mathbf{P_{k-1}})\Delta_k x + \frac{\partial F}{\partial t}(\mathbf{P_{k-1}})\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\mathbf{P_{k-1}})[\Delta_k x]^2$$

This intuitive line of reasoning leads us to

**Ito's Lemma** Let $F(x_t, t)$ be a twice differentiable function of $t$ and a random process $x_t$ governed by an SDE of the form $dx_t = a(x_t, t)dt + \sigma(x_t, t)dW_t$ where $a(x_t, t)$ and $\sigma(x_t, t)$ are square-integrable and $W_t$ an $I_t$-adapted Wiener process. Then $F(x_t, t)$ satisfies the SDE

$$dF_t = \frac{\partial F}{\partial x}dx_t + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}[dx_t]^2.$$

(Equality holds in the sense of mean square limits.)

**Exercise**

**3.7** Decide whether each term is negligible as $\Delta t \to 0$:

**a)** $\frac{\partial F}{\partial x}(\mathbf{P_{k-1}})\Delta_k x$

**b)** $\frac{\partial F}{\partial t}(\mathbf{P_{k-1}})\Delta t$

**c)** $\frac{\partial^2 F}{\partial x \partial t}(\mathbf{P_{k-1}})\Delta_k x \Delta t$

## 3.8 Utilizing Ito's Lemma

Ito's Lemma may be utilized in two important ways:

- determing an SDE satisfied by a given stochastic process $F(x_t, t)$; and
- obtaining certain Ito integrals.

For example, if $F(x_t, t) = \frac{1}{2}x_t^2$, where $x_t$ is a process satisfying the hypothesis of Ito's lemma, then $F$ satisfies the SDE

$$dF_t = x_t dx_t + \frac{1}{2}dt.$$

The corresponding Ito integral equation is

$$F(x_T, T) = \int_0^T x_t dx_t + T/2.$$

Substituting for $F$ and re-arranging terms, we obtain

$$\int_0^T x_t dx_t = \frac{1}{2}x_T^2 - T/2.$$

(In section 3.5, we verified this last equality using the definition of the Ito integral and a much lengthier calculation.)

**Exercise**

**3.8** Let $S(W_t, t) = x_0 e^{[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t]}$ where $\mu$ and $\sigma$ are positive real constants and $W_t$ a stochastic process which satisies the hypothesis of Ito's Lemmma. Find an SDE for $S$.

# 4 The Black-Scholes Equation

The options price model intoduced by Fischer Black and Myron Scholes [1973] led to Scholes and Robert Merton receiving the Nobel prize in economics in 1997 (Black was not alive to be a co-recipient.) The Black-Scholes model is a partial differentation equation satisfied by the call option price $F = F(x_t, t)$ considered as a twice-differentiable deterministic function of the stock price $x_t$ (which does not pay dividends) and time $t$ $(0 \leq t \leq T_{exp})$. The stock price satisfies the SDE

$$dx = \sigma x dW + \mu x dt. \tag{3}$$

where $\mu$ and $\sigma$ are positive constants and $W$ a well-behaved Wiener process. (For simplicity of notation, we have omitted the subscript $t$).

The value of the option at time at time of expiry $T_{exp}$ is

$$F(x_{T_{exp}}, T_{exp}) = max[x_{T_{exp}} - K, 0], \tag{4}$$

where $K$ is the strike price (see Section 2.2).

By Ito's Lemma, we know that the option price $F$ satisifes the SDE

$$dF = \frac{\partial F}{\partial x}[\sigma x dW + \mu x dt] + \frac{\partial F}{\partial t}dt + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}[dx]^2.$$

Applying the SDE relation $(dx)^2 = dt$ (see **Exercise 3.5b**), we have

$$dF = [\mu x \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}]dt + \sigma x \frac{\partial F}{\partial x} dW. \tag{5}$$

Note that by equations (3) and (5), randomness in the change in both the stock price $dx$ and option price $dF$ is due to the same term $dW$. Employing the Delta hedging strategy introduced in Section 3.1 and **Exercise 3.1**, we are able to create

[Neftci 1996] a risk-free portfolio with value $P$ by purchasing 1 unit of options and selling short $\frac{\partial F}{\partial x}$ units of stock ($P = F - x\frac{\partial F}{\partial x}$) so that

$$dP = dF - \frac{\partial F}{\partial x}dx.$$

Using (3) and (5), we have (**Exercise 4.1a**)

$$dP = \frac{\partial F}{\partial t}dt + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}dt. \tag{6}$$

Since this investment is risk-neutral, in order to avoid arbitrage, if the risk-free interest rate is $r$, we must have

$$\frac{dP}{dt} = rP \Rightarrow$$
$$dP = rPdt.$$

By substituting for $P$ and $dP$, and cancelling $dt$, (**Exercise 4.1b**) we obtain

**Black-Scholes Equation**

$$-rF + r\frac{\partial F}{\partial x}x_t + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} = 0. \tag{7}$$

Using the boundary condition (4), Black and Scholes obtained the following exact solution for (7) on the interval $t \in [0, T_{exp}]$ (**Exercise 4.1c**):

$$F(x,t) = xN(d_1) - Ke^{-r(T_{exp}-t)}N(d_2) \tag{8}$$

where

$$d_1 = \frac{ln(x/K) + (r + \frac{1}{2}\sigma^2)(T_{exp} - t)}{\sigma\sqrt{T_{exp} - t}},$$

$$d_2 = d_1 - \sigma\sqrt{T_{exp} - t},$$

and for $i = 1, 2$

$$N(d_i) = \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du.$$

A major strength of the Black-Scholes model is its ability to give a closed-form solution which shows how the no-arbitrage European-call option price $F(x,t)$ depends on

$t$ = time;

$x$ = stock price at time $t$;

$K$ =strike price;

$T_{exp}$ = time of expiration;

$\sigma$ = "volatility" parameter;

$r$ = risk free interest rate.

As such, one can determine how the option price is affected by one or more changes in the parameters. For example, the price is an increasing function in the stike price $K$, time of expiration $T_{exp}$, volatility $\sigma$ and interest rate $r$.

The basic Black-Scholes model assumes that

- the underlying asset is a stock which does not pay dividends;
- the risk-free interest rate is constant; and
- there are no transaction costs.

In practice, one or more of these assumptions usually does not hold. (Refinements to the model which account for more realistic assumptions have been extensively studied, but are beyond the scope of this Module.) Even so, the basic model is widely used by practioners and real world option prices are remarkably close to those predicted by the Black-Scholes equation.

**Exercise**

**4.1 a)** Show how to obtain equation (6).

**b)** Show how to obtain equation (7).

**c)** Verify that (8) satifies (7) and (4).

# 5 Further Study

In this Module we have introduced the concepts of arbitrage (Section 2) and stochastic calculus (Section 3) used to obtain the Black-Scholes partial differential equation for the option price as a function of time and the underlying stock price (Section 4). We recommend the books by Etheridge[2002], Ross [2003] and Neftci [1996] and the paper by Black and Scholes [1973] for a more complete treatment of this material. Wilmott [2001] is a user-friendly introduction to math finance written at an easier mathematical level, while Mun [2002] is a practitioner's handbook and valuable reference for those interested in further study.

# 6 Solutions to Selected Exercises

**2.1.1** $x = A_1\beta + A_2\beta^2 + \cdots + A_n\beta^n$ where $\beta = (1+r)^{-1}$.

**2.1.2 a)** $A(1+r^*) = A_1 \Rightarrow r^* = \frac{A_1}{A} - 1$.

**b)** $x = A_1\beta + A_2\beta^2 \Rightarrow A_2\beta^2 + A_1\beta - x = 0$.

**2.1.3 a)** $A = \lim_{n\to\infty} A_0[1 + (r/n)]^{nt} = A_0 e^{rt}$. To evaluate the limit, observe that

$$\begin{aligned}
\lim_{n\to\infty} A_0(1+\frac{r}{n})^{nt} &= A_0 \lim_{n\to\infty}(1+\frac{r}{n})^{nt} \\
&= A_0 \lim_{n\to\infty} e^{\ln(1+\frac{r}{n})^{nt}} \\
&= A_0 e^{\lim_{n\to\infty}\ln(1+\frac{r}{n})^{nt}} \\
&= A_0 e^{rt}.
\end{aligned}$$

The last limit in the exponent of the exponential function is evaluated using L'Hopital's Rule:

$$\begin{aligned}
\lim_{n\to\infty} nt\ln(1+\frac{r}{n}) &= t \lim_{n\to\infty} \frac{\ln(1+\frac{r}{n})}{\frac{1}{n}} \\
&= t \lim_{n\to\infty} \frac{(\frac{-r}{n^2})(\frac{1}{1+\frac{r}{n}})}{\frac{-1}{n^2}} \\
&= rt \lim_{n\to\infty} \frac{1}{1+\frac{r}{n}} \\
&= rt.
\end{aligned}$$

**b)** $x = ye^{-rt}$.

**2.2.1** 55/3.

**2.2.2 a)** By selling short 1/3 share, you obtain initially 100/3. Of this amount, investing $C$ in the call option leaves $A_0 = 100/3 - C > \frac{50}{3(1+r)}$ to be deposited in the bank. If at $t = 1$ the stock price is 200, your revenue from the call option is 50. Together with your bank deposit which has grown to the amount $A_0(1+r) > 50/3$, after spending 200/3 for the 1/3 share of stock you sold short, you achieve a postive gain in the amount of $A_0(1+r) - 50/3$.

On the other hand, if the $t = 1$ stock price drops to 50, you will still have $A_0(1+r) > 50/3$ in savings, will receive no revenue from your call option, and must spend 50/3 for the stock you sold short. In this case, you are again guaranteed the same amount of profit, namely $A_0(1+r) - 50/3$.

**b)** At $t = 0$, sell one unit of call options, borrow $\frac{50}{3(1+r)}$ from the bank, and use 100/3 to purchase 1/3 share of stock. Deposit the amount $C + \frac{50}{3(1+r)} - 100/3$ into a savings account.

At $t = 1$, if the stock goes up to 200, your 1/3 share of stock plus savings with interest total $[C + \frac{50}{3(1+r)} - 100/3](1+r) + 200/3$. To repay the bank loan with interest and call obligation, you must spend 50/3 + 50=200/3. Thus you are left with a profit of $[C + \frac{50}{3(1+r)} - 100/3](1+r) > 0$ .

On the other hand, if at $t = 1$, the stock price goes down, your 1/3 share of stock plus savings with interest total $[C + \frac{50}{3(1+r)} - 100/3](1+r) + 50/3$ and your payment obligations total 50/3, again leaving you with a positive gain in the amount $[C + \frac{50}{3(1+r)} - 100/3](1+r) > 0$.

**2.2.3** No. We claim it is better to sell the stock short and wait until time of expiry to utilize an option rather than exercising the option before time of expiry. To see this, suppose the stock price rises to $P_1 > K$ at time $T_1 < T_{exp}$. The American call option allows you to realize at $T_1$ a gain of $P_1 - K$, which grows by interest to $\Gamma = (P_1 - K)(1+r)^{T_{exp}-T_1}$ at time $T_{exp}$. If, on the other hand, you sell the stock short at time $T_1$, at time of expiry, if the stock price is $P_{exp}$, you will have either

- $P_1(1+r)^{T_{exp}-T_1} - P_{exp} \quad > \Gamma$ if $P_{exp} \leq K$ (option is worthless); or
- $P_1(1+r)^{T_{exp}-T_1} - K \quad > \Gamma$ if $P_{exp} > K$ (option is utilized).

Observe that in both cases, selling the stock is preferable to exercising a call option before time of expiry.

**2.2.4** With the money we obtain from selling one unit of options, we create a portfolio of $x_1$ dollars and $x_2$ units of stock. The value of the option is $V(x_1, x_2) =$

$x_1 + x_2P_0$. To hedge the contingent claim made at the end of the period, we require

$$\begin{aligned} x_1(1+r) + P_0ux_2 &= (P_0u - P_{claim})_+ \\ x_1(1+r) + P_0dx_2 &= (P_0d - P_{claim})_+. \end{aligned}$$

Denoting the solution to this system as $(x_1^*, x_2^*)$, the value of the option, $V(x_1^*, x_2^*)$ may be shown by elementary algebra to equal the stated amount.

**2.3** Suppose $\Delta = S + P - C < Ke^{-rT_{exp}}$. At $t = 0$ we borrow an amount $\Delta$ from the bank with which we buy one share of stock, buy one put option, and sell one call option. At time $T_{exp}$, if the stock price $S_{exp}$ is less than or equal to $K$, the call option we sold is worthless and we can exercise our put option and so realize at time $T_{exp}$ a positive gain $K - \Delta e^{rT_{exp}}$. If the stock price $S_{exp}$ is greater than $K$, our put option is worthless and the call option will require us to sell our stock at strike price $K$. Hence at time $T_{exp}$ we will again have positive gain $K - \Delta e^{rT_{exp}}$.

**2.4** **a)** Since $X_i = 1$ if the stock goes up in price during $(i-1)^{st}$ period (and 0 otherwise), $Y$ gives the total number of periods in which the stock goes up in price.

**b)** $V = S_n - K$ (if $S_n > K$) and $V = 0$ otherwise.

**c)** $C = (1+r)^{-n}E[V]$.

**3.1** If (instantaneously) the stock price increases by an amount $dx_0$, the short position in stock loses by an amount $g'(x_0)dx_0$. The value of the one unit of call option will, however, gain by $dg \approx g'(x_0)dx_0$.

**3.3.1** **a)** $E_s[x_t - x_s] = E_s[x_t] - E_s[x_s] = x_s - x_s = 0$.

**b)** In general, the price of a stock is expected to increase over a long period of time. That is, if $0 \leq s << t \leq T$, then $E_s[x_s] < E_s[x_t]$ or $E_s[x_t - x_s] > 0$, in violation of a).

**3.3.2** **a)** By definition, $x_t - x_0$ is normally distributed with mean zero and variance $t$. Thus, $t = VAR[x_t] = E[x_t^2] - (E[x_t])^2 = E[x_t^2]$.

**(b)** Define $U = \frac{\Delta_{i+1}x}{\sqrt{t_{i+1} - t_i}}$ so that $E[(\Delta_{i+1}x)^4] = (t_{i+1} - t_i)^2 E(U^4)$. Observe that $U$ is a standard unit normal random variable with generating function $\phi(t) = e^{\frac{t^2}{2}}$. The result follows since $E[U^4] = \phi''''(0) = 3$.

**3.4** First note that

$$
\begin{aligned}
(\sum_{i=0}^{n-1} a_i^2)^2 &= (a_0^2 + a_1^2 + \cdots + a_{n-1}^2)^2 \\
&= (a_0^4 + a_1^4 + \cdots + a_{n-1}^4) + \\
&\quad (2a_0^2a_1^2 + 2a_0^2a_2^2 + \cdots + 2a_0^2a_{n-1}^2) + \\
&\quad (2a_1^2a_2^2 + 2a_1^2a_3^2 + \cdots + 2a_1^2a_{n-1}^2) + \cdots + \\
&\quad 2a_{n-2}^2a_{n-1}^2 \Rightarrow \\
(\sum_{i=0}^{n-1} a_i^2)^2 &= \sum_{i=0}^{n-1} a_i^4 + 2\sum_{i=0}^{n-2}\sum_{i<j}^{n-1} a_i^2a_j^2. \qquad (9)
\end{aligned}
$$

Using (9), the fact that $E[V_n] = T$, and the properties of Wiener processes given in Section 3.3, we have

$$
\begin{aligned}
E[(V_n - T)^2] &= \\
E[V_n^2 - 2TV_n + T^2] &= \\
E[\sum_{i=0}^{n-1}(\Delta_{i+1}x)^4 + 2\sum_{i=0}^{n-2}\sum_{i<j}^{n-1}(\Delta_{i+1}x)^2(\Delta_{j+1}x)^2 - 2TV_n + T^2] &= \\
3n(\frac{T}{n})^2 + 2\frac{T}{n}[\frac{T}{n}\frac{n(n-1)}{2}] - 2T^2 + T^2 &= \\
3\frac{T^2}{n} + \frac{T^2(n-1)}{n} - T^2 &= \\
2\frac{T^2}{n}
\end{aligned}
$$

**3.5** Both **a)** and **b)** are equivalent to the result stated as Lemma 2 (Section 3.4).

**3.7** Only **c)** is negligible. (Use the fact that $E[(\sigma_k W_k)^2] = \sigma_k^2 \Delta t$.)

**3.8** $dS = \sigma S dW + (\mu - \frac{1}{2}\sigma^2)Sdt + \frac{1}{2}(\sigma)^2 S(dW)^2 = \sigma SdW + \mu Sdt$. (Note the use of Exercise 3.5b).)

**4.1** **a)**

$$
dP = dF - \frac{\partial F}{\partial x}dx
$$

$$
\begin{aligned}
&= [\mu x \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}]dt + \sigma x \frac{\partial F}{\partial x} dW \\
&\quad -\frac{\partial F}{\partial x}(\mu x dt + \sigma x dW) \\
&= \frac{\partial F}{\partial t} dt + [\frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}]dt.
\end{aligned}
$$

**b)**

$$
\begin{aligned}
dP &= rPdt \Rightarrow \\
\frac{\partial F}{\partial t} dt + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} dt = r(F - x\frac{\partial F}{\partial x})dt \Rightarrow \\
-rF + rx\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} &= 0.
\end{aligned}
$$

**c)** See Ross[2003]

# 7 References


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