DENDRITES AND SYMMETRIC PRODUCTS

GERARDO ACOSTA, RODRIGO HERNÁNDEZ-GUTIÉRREZ,
AND VERÓNICA MARTÍNEZ-DE-LA-VEGA

Abstract. For a given continuum X and a natural number n, we consider the hyperspace $F_n(X)$ of all nonempty subsets of X with at most n points, metrized by the Hausdorff metric. In this paper we show that if X is a dendrite whose set of end points is closed, $n \in \mathbb{N}$ and Y is a continuum such that the hyperspaces $F_n(X)$ and $F_n(Y)$ are homeomorphic, then Y is a dendrite whose set of end points is closed.

1. Introduction

A *continuum* is a nondegenerate, compact, connected metric space. Let $\mathbb{N}$ represent the set of positive integers. For a given continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

$$F_n(X) = \{A \subset X\colon A \text{ is nonempty and has at most } n \text{ points}\}$$

and

$$C_n(X) = \{A \subset X\colon A \text{ is closed, nonempty and has at most } n \text{ components}\}.$$

We call $C_n(X)$ the *n-fold hyperspace of* X and $F_n(X)$ the *n-th symmetric product of* X. Both $F_n(X)$ and $C_n(X)$ are metrized by the Hausdorff metric H ([17, Definition 2.1]).

If two continua X and Y are homeomorphic, we write $X \approx Y$. Note that if X and Y are continua, then $X \approx Y$ if and only if $F_1(X) \approx F_1(Y)$. Let $\mathcal{G}$ be a class of continua, $n \in \mathbb{N}$ and $X \in \mathcal{G}$. We say that X has *unique hyperspace* $F_n(X)$ *in* $\mathcal{G}$ if whenever $Y \in \mathcal{G}$ is such that $F_n(X) \approx F_n(Y)$, it follows that $X \approx Y$. Similarly, X has *unique hyperspace* $C_n(X)$ *in* $\mathcal{G}$ if whenever $Y \in \mathcal{G}$ is such that $C_n(X) \approx C_n(Y)$, we have $X \approx Y$. If $\mathcal{G}$ is the class of all continua, we simply say that X has unique hyperspace $F_n(X)$ or unique hyperspace $C_n(X)$, respectively. Note that each continuum X has unique hyperspace $F_1(X)$.

A *dendrite* is a locally connected continuum that contains no simple closed curves. Throughout this paper we denote by $\mathcal{D}$ the class of dendrites whose set of end points is closed. In [11, Theorem 10] it is shown that if $X \in \mathcal{D}$ is not an arc, then X has unique hyperspace $C_1(X)$. In [16, Theorem 3]

Date: August 15, 2008.

2000 *Mathematics Subject Classification.* 54B20, 54C15, 54F15, 54F50.

Key words and phrases. Continuum, contractibility, dendrite, finite graph, unique hyperspace.

that every $X \in \mathcal{D}$ has unique hyperspace $C_2(X)$, and in [12, Theorem 5.7] that each $X \in \mathcal{D}$ has also unique hyperspace $C_n(X)$, for $n \geq 3$. In [2, Theorem 5.1] it is proved that if the set of end points of the dendrite Y is not closed, then Y does not have unique hyperspace $C_1(Y)$ in the class of dendrites. This result is not known for $n \geq 2$. By [1, Lemma 11] an arc Y has unique hyperspace $C_1(Y)$ in the class of dendrites, but not in the class of all continua.

In the First Workshop in Hyperspaces and Continuum Theory, celebrated in the city of Puebla, Mexico, July 2-13, 2007, the problem to determine if every element $X \in \mathcal{D}$ has unique hyperspace $F_n(X)$ was asked by A. Illanes. During the workshop, the three authors of this paper showed that if $X \in \mathcal{D}$, $n \in \mathbb{N}$ and Y is a continuum such that $F_n(X) \approx F_n(Y)$, then $Y \in \mathcal{D}$. This is the main result of this paper. In the same workshop, D. Herrera-Carrasco, M. de J. López and F. Macías-Romero proved that every element $X \in \mathcal{D}$ has unique hyperspace $F_n(X)$ in $\mathcal{D}$ ([13, Theorem 3.5]). Combining these results it follows that every element $X \in \mathcal{D}$ has unique hyperspace $F_n(X)$. This is a partial positive answer to the following problem, which remains open.

Question 1.1. *Let* X *be a dendrite and* $n \in \mathbb{N} - \{1\}$. *Does* X *have unique hyperspace* $F_n(X)$*?*

2. General Notions and Facts

All spaces considered in this paper are assumed to be metric. For a space X, a point $x \in X$ and a positive number ε, we denote by $B_X(x, \varepsilon)$ the open ball in X centered at x and having radius ε. If A is a subset of the space X, we use the symbols $\mathrm{cl}_X(A)$, $\mathrm{int}_X(A)$ and $\mathrm{bd}_X(A)$ to denote the closure, the interior and the boundary of A in X, respectively. We denote the diameter of A by $\mathrm{diam}\,(A)$, and the cardinality of A by $|A|$. The letter I stands for the unit interval $[0, 1]$ in the real line $\mathbb{R}$.

A *finite graph* is a continuum that can be written as the union of finitely many arcs, each two of which intersect in a finite set. A *tree* is a finite graph that contains no simple closed curves.

If X is a continuum, $U_1, U_2, \ldots, U_m \subset X$ and $n \in \mathbb{N}$ we define:

$$\langle U_1, U_2, \ldots, U_m \rangle_n = \left\{ A \in F_n(X) \colon A \subset \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \right\}.$$

It is known that the sets of the form $\langle U_1, U_2, \ldots, U_m \rangle_n$, where $m \in \mathbb{N}$ and $U_1, U_2, \ldots, U_m$ are open in X, form a basis of the topology of $F_n(X)$, i.e., a basis for the topology induced by the Hausdorff metric H on $F_n(X)$ ([17, Theorems 1.2 and 3.1]).

If $n \in \mathbb{N}$, then an *n-cell* is a space homeomorphic to the Cartesian product I^n.

Theorem 2.1. *Let* X *be a continuum and* $n \in \mathbb{N}$. *Given* $i \in \{1, 2, \ldots, n\}$ *let* J_i *be an arc in* X *with end points* a_i *and* b_i. *If the sets* $J_1, J_2, \ldots, J_n$ *are* *pairwise disjoint, then* $\langle J_1, J_2, \ldots, J_n\rangle_n$ *is an* n*-cell in* $F_n(X)$ *whose manifold interior is the set* $\langle J_1 - \{a_1, b_1\}, \ldots, J_n - \{a_n, b_n\}\rangle_n$.

Proof. Given $(x_1, x_2, \ldots, x_n) \in J_1 \times J_2 \times \cdots \times J_n$, let $g(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\}$. It is easy to see that $g\colon J_1 \times J_2 \times \cdots \times J_n \to \langle J_1, J_2, \ldots, J_n\rangle_n$ is a homeomorphism whose restriction to $\prod_{i=1}^{n} (J_i - \{a_i, b_i\})$ is a homeomorphism from $\prod_{i=1}^{n} (J_i - \{a_i, b_i\})$ onto $\langle J_1 - \{a_1, b_1\}, \ldots, J_n - \{a_n, b_n\}\rangle_n$. □

From now on, in this section, the letter X represents a dendrite. For properties of dendrites we refer the reader to [21, Chapter 10]. If $p \in X$ then by the *order of* p *in* X, denoted by $\text{ord}_p X$, we mean the Menger-Urysohn order (see [21, Definition 9.3] and [23, (1.1), (iv), p. 88]). We say that $p \in X$ is an *end point of* X if $\text{ord}_p X = 1$. The set of all such points is denoted by $E(X)$. Let

$$\begin{aligned} E_a(X) \;\; = \;\; &\{p \in E(X)\colon \text{ there is a sequence in } E(X) - \{p\} \\ &\text{ that converges to } p\}. \end{aligned}$$

If $p \in E(X) - E_a(X)$ we call p an *isolated end point of* X. If $\text{ord}_p X = 2$ we say that p is an *ordinary point of* X. The set of all such points is denoted by $O(X)$. By [18, Theorem 8, p. 302], $O(X)$ is dense in X. If $\text{ord}_p X \geq 3$, we say that p is a *ramification point of* X. The set of all such points is denoted by $R(X)$.

The following result is easy to prove.

Theorem 2.2. *Let* X *be a dendrite and* $n \in \mathbb{N}$. *Assume that* $A \in F_n(X)$ *and that* $\mathcal{U}$ *is a neighborhood of* A *in* $F_n(X)$. *Then, for each* $k \in \mathbb{N}$ *with* $|A| \leq k \leq n$, *there is* $C \subset O(X)$ *such that* $|C| = k$ *and* $C \in \mathcal{U}$.

If $p, q \in X$ and $p \neq q$, then there is only one arc in X joining p and q. We denote such arc by $[p, q]$. We also consider the sets $(p, q) = [p, q] - \{p, q\}$, $[p, q) = [p, q] - \{q\}$ and $(p, q] = [p, q] - \{p\}$. Let $[p, q]$ be an arc in X such that $(p, q) \subset O(X)$. We say that $[p, q]$ is:

a) *internal* if $p, q \in R(X)$;
b) *external* if one end point of $[p, q]$ is an end point of X, and the other end point of $[p, q]$ is a ramification point of X.

Note that if $[p, q]$ is an internal arc in X, then $\text{int}_X([p, q]) = (p, q)$. If $[p, q]$ is an external arc in X and $p \in E(X)$, then $\text{int}_X([p, q]) = [p, q)$.

Given $n \in \mathbb{N}$, we consider the following subsets of $F_n(X)$:

$$EA_n(X) = \{A \in F_n(X)\colon A \cap E_a(X) \neq \emptyset\},$$

$$R_n(X) = \{A \in F_n(X)\colon A \cap R(X) \neq \emptyset\}$$

and

$$\Lambda_n(X) = F_n(X) - (R_n(X) \cup EA_n(X)).$$

Note that $A \in \Lambda_n(X)$ if and only if $A \in F_n(X)$ and A is contained in $O(X) \cup (E(X) - E_a(X))$.

3. The class $\mathcal{D}$

Recall that $\mathcal{D}$ is the class of all dendrites whose set of end points is closed. Let $X \in \mathcal{D}$. By [3, Theorem 3.3] the order of every point of X is finite. Let us assume that $s \in X$ is the limit of a sequence $(s_n)_n$ of distinct ramification points of X and that $s \neq s_1$. By [3, Proposition 3.4] s is both the limit of a sequence of ramification points of X, all in the arc $[s, s_1]$, and the limit of a sequence of end points, all different than s. Now assume that $e \in X$ is the limit of a sequence $(e_n)_n$ of distinct end points of X and that $e \neq e_1$. Then e is also the limit of a sequence of ramification points of X, all in the arc $[e, e_1]$. Hence $e \in E_a(X)$ if and only if e is the limit of a sequence of ramification points of X.

Theorem 3.1. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. If $A \in F_n(X) - EA_n(X)$, then there exists a tree T in X such that:*

$(*)$ *$A \subset \mathrm{int}_X(T)$ and $T \cap E_a(X) = \emptyset$.*

Proof. Let $A \in F_n(X) - EA_n(X)$. We proceed by induction over $|A|$. If $|A| = 1$, then $A = \{x\}$. Let $k = \mathrm{ord}_x X$. Since $X \in \mathcal{D}$, k is finite, so $X - \{x\}$ has exactly k components $C_1, C_2, \ldots, C_k$. Since $x \notin E_a(X)$, for each $i = \{1, 2, \ldots, k\}$, there is $p_i \in O(X) \cap C_i$ such that if $T = \bigcup_{i=1}^{k} [p_i, x]$, then $T - \{x\} \subset O(X)$. Hence T is a tree in X that satisfies $(*)$.

Now suppose that if $B \in F_n(X) - EA_n(X)$ contains i points, with $i < n$, then there is a tree G in X that satisfies $(*)$, replacing A by B and T by G, respectively. Assume that $|A| = i + 1$ and let $A = \{x_1, x_2, \ldots, x_{i+1}\}$. Let T_1 be a tree in X such that $A - \{x_{i+1}\} \subset \mathrm{int}_X(T_1)$ and $T_1 \cap E_a(X) = \emptyset$. By the first part of this proof, there exists a tree T_2 in X such that $x_{i+1} \in \mathrm{int}_X(T_2)$ and $T_2 \cap E_a(X) = \emptyset$. Thus $T = T_1 \cup T_2 \cup [x_i, x_{i+1}]$ is a tree in X that satisfies $(*)$. □

Theorem 3.2. *Let $X \in \mathcal{D}$ and $m, n \in \mathbb{N}$ so that $m \leq n$. Let $\mathcal{U} = \langle U_1, U_2, \ldots, U_m \rangle_n$ be an open subset of $F_n(X)$ such that:*

1) *U_i is an open connected subset of X, for each $i \in \{1, 2, \ldots, m\}$;*
2) *$U_i \cap U_j = \emptyset$ if $i, j \in \{1, 2, \ldots, m\}$ and $i \neq j$.*

For each $i \in \{1, 2, \ldots, m\}$, let $\{J^i_\alpha \colon \alpha \in \mathcal{A}_i\}$ be the set of components of $U_i \cap [O(X) \cup (E(X) - E_a(X))]$. Then the components of $\mathcal{U} \cap \Lambda_n(X)$ are the nonempty sets of the form:

$$\left\langle J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k} \right\rangle_n,$$

where $\{r_1, r_2, \ldots, r_k\} = \{1, 2, \ldots, m\}$, the sets $J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k}$ are pairwise different and $\alpha_t \in \mathcal{A}_{r_t}$, for every $t \in \{1, 2, \ldots, k\}$.

Proof. It is easy to see that for each $i \in \{1, 2, \ldots, m\}$ and every $\alpha \in \mathcal{A}_i$, J^i_α is an open connected subset of X. Let $J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k}$ be a finite collection of pairwise different sets such that $\{r_1, r_2, \ldots, r_k\} = \{1, 2, \ldots, m\}$ and $\alpha_t \in \mathcal{A}_{r_t}$, for every $t \in \{1, 2, \ldots, k\}$. Since the sets $J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k}$ are open and connected, by [19, Lemma 1],

$$\left\langle J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k} \right\rangle_n$$

is an open connected subset of $F_n(X)$. Let $J^{s_1}_{\epsilon_1}, J^{s_2}_{\epsilon_2}, \ldots, J^{s_l}_{\epsilon_l}$ be a finite collection of pairwise different sets such that: $\{s_1, s_2, \ldots, s_l\} = \{1, 2, \ldots, m\}$, $\epsilon_v \in \mathcal{A}_{s_v}$, for every $v \in \{1, 2, \ldots, l\}$, and

$$\left\{J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k}\right\} \neq \left\{J^{s_1}_{\epsilon_1}, J^{s_2}_{\epsilon_2}, \ldots, J^{s_l}_{\epsilon_l}\right\}.$$

It is not difficult to see that:

$$\left\langle J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k} \right\rangle_n \cap \left\langle J^{s_1}_{\epsilon_1}, J^{s_2}_{\epsilon_2}, \ldots, J^{s_l}_{\epsilon_l} \right\rangle_n = \emptyset.$$

Now assume that $\mathcal{C}$ is a component of $\mathcal{U} \cap \Lambda_n(X)$. Note that, for every $A \in \mathcal{C}$, there is a unique finite collection

$$J^{s_1}_{\sigma_1}, J^{s_2}_{\sigma_2}, \ldots, J^{s_w}_{\sigma_w}$$

of pairwise different sets such that: $\{s_1, s_2, \ldots, s_w\} = \{1, 2, \ldots, m\}$, $\sigma_j \in \mathcal{A}_{s_j}$, for each $j \in \{1, 2, \ldots, w\}$, and

$$A \in \mathcal{V}_A = \left\langle J^{s_1}_{\sigma_1}, J^{s_2}_{\sigma_2}, \ldots, J^{s_w}_{\sigma_w} \right\rangle_n.$$

Hence $\mathcal{C} = \bigcup_{A \in \mathcal{C}} \mathcal{V}_A$, which expresses the open connected set $\mathcal{C}$ as a union of nonempty pairwise disjoint open connected sets. Thus $\mathcal{C}$ is of the form $\left\langle J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k} \right\rangle_n$ where $\{r_1, r_2, \ldots, r_k\} = \{1, 2, \ldots, m\}$, the sets $J^{r_1}_{\alpha_1}, J^{r_2}_{\alpha_2}, \ldots, J^{r_k}_{\alpha_k}$ are pairwise different and $\alpha_t \in \mathcal{A}_{r_t}$, for every $t \in \{1, 2, \ldots, k\}$. □

Assume that $X \in \mathcal{D}$. It is not difficult to see that $\Lambda_n(X)$ is an open subset of $F_n(X)$. As a particular case of Theorem 3.2 we obtain the following result, which is the equivalent version of [7, Lemma 4.1] for elements of $\mathcal{D}$.

Theorem 3.3. *Let $X \in \mathcal{D}$ such that X is not an arc and $n \in \mathbb{N}$. Then the components of $\Lambda_n(X)$ are exactly the sets of the form:*

$$\left\langle \operatorname{int}_X(I_1), \operatorname{int}_X(I_2), \ldots, \operatorname{int}_X(I_m) \right\rangle_n,$$

where $m \leq n$, I_j is either an internal or an external arc in X for every $j \in \{1, 2, \ldots, m\}$, and the sets $\operatorname{int}_X(I_1), \operatorname{int}_X(I_2), \ldots, \operatorname{int}_X(I_m)$ are pairwise disjoint.

The following result is the equivalent version of [7, Lemma 4.3], for elements of $\mathcal{D}$.

Theorem 3.4. *Let $X \in \mathcal{D}$ and $n \geq 4$. If $A \in F_{n-1}(X)$, then no neighborhood of A in $F_n(X)$ can be embedded in $\mathbb{R}^n$.*

Proof. We show first that:

(∗) if $C \in F_{n-1}(X) - EA_n(X)$, then no neighborhood of C in $F_n(X)$ can be embedded in $\mathbb{R}^n$.

To show $(*)$ let $C \in F_{n-1}(X) - EA_n(X)$ and assume that there is a neighborhood $\mathcal{V}$ of C in $F_n(X)$ that can be embedded in $\mathbb{R}^n$. By Theorem 3.1, there is a tree T in X such that $C \subset \operatorname{int}_X(T)$ and $T \cap E_a(X) = \emptyset$. Then $\mathcal{V} \cap F_n(T)$ is a neighborhood of C in $F_n(T)$ that can be embedded in $\mathbb{R}^n$. Since this contradicts [7, Lemma 4.3], claim $(*)$ holds.

To show the theorem let $A \in F_{n-1}(X)$. Assume that there is a neighborhood $\mathcal{U}$ of A in $F_n(X)$ that can be embedded in $\mathbb{R}^n$. By Theorem 2.2, there is $C \subset O(X)$ such that $|C| = |A|$ and $C \in \operatorname{int}_{F_n(X)}(\mathcal{U})$. Since $A \in F_{n-1}(X)$ it follows that $C \in F_{n-1}(X) - EA_n(X)$. Then, by $(*)$, no neighborhood of C in $F_n(X)$ can be embedded in $\mathbb{R}^n$. However, since $C \in \operatorname{int}_{F_n(X)}(\mathcal{U})$, the set $\mathcal{U}$ is a neighborhood of C in $F_n(X)$ that can be embedded in $\mathbb{R}^n$. This contradiction completes the proof of the theorem. □

4. THE SET $\mathcal{E}_n(X)$

Given a continuum X and a natural number n, we consider the following set:

$$\mathcal{E}_n(X) = \{A \in F_n(X)\colon A \text{ has a neighborhood in } F_n(X) \text{ which is an } n\text{-cell}\}.$$

In this section we prove some properties of $\mathcal{E}_n(X)$.

Theorem 4.1. *Let X and Y be continua and $n \in \mathbb{N}$. If $h\colon F_n(X) \to F_n(Y)$ is a homeomorphism, then $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$.*

A *simple triod* is a continuum G that can be written as the union of three arcs I_1, I_2 and I_3 such that: $I_1 \cap I_2 \cap I_3 = \{p\}$, p is an end point of each arc I_i and $(I_i - \{p\}) \cap (I_j - \{p\}) = \emptyset$, if $i \neq j$. The point p is called the *core of* G.

Given a continuum X let:

$$T(X) = \{p \in X\colon p \text{ is the core of a simple triod in } X\}.$$

Let X be a locally connected continuum and $A \in \mathcal{E}_n(X)$. In [7, Lemma 3.1] it is shown that $A \cap T(X) = \emptyset$. A straightforward modification can be applied to obtain the following result.

Theorem 4.2. *Let X be a locally connected continuum and $n \in \mathbb{N}$. If $A \in \mathcal{E}_n(X)$, then $A \cap \operatorname{cl}_X(T(X)) = \emptyset$.*

Theorem 4.3. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. Then $\Lambda_n(X) - F_{n-1}(X) \subset \mathcal{E}_n(X)$.*

Proof. Take $A \in \Lambda_n(X) - F_{n-1}(X)$. Then $|A| = n$, we write

$$A = \{x_1, x_2, \ldots, x_n\}.$$

Since $A \in \Lambda_n(X)$, we have $A \subset O(X) \cup (E(X) - E_a(X))$. Then there exist n pairwise disjoint arcs $J_1, J_2, \ldots, J_n$ in X such that $x_i \in \operatorname{int}_X(J_i)$, for each $i \in \{1, 2, \ldots, n\}$, and

$$J_i \cup J_2 \cup \cdots \cup J_n \subset O(X) \cup (E(X) - E_a(X)).$$

Note that $\langle J_1, J_2, \ldots, J_n\rangle_n$ is a neighborhood of A in $F_n(X)$ which is an n-cell, by Theorem 2.1. Then $A \in \mathcal{E}_n(X)$. □

Theorem 4.4. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. Then $\mathcal{E}_n(X)$ is dense in $F_n(X)$.*

Proof. Let $\mathcal{U}$ be a nonempty open subset of $F_n(X)$. By Theorem 2.2 there is $D \subset O(X)$ such that $|D| = n$ and $D \in \mathcal{U}$. Note that $D \in \Lambda_n(X) - F_{n-1}(X)$ so, by Theorem 4.3, $D \in \mathcal{E}_n(X)$. This shows that $\mathcal{E}_n(X)$ is dense in $F_n(X)$. □

Theorem 4.5. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. Then*

a) $\mathcal{E}_n(X) \subset \Lambda_n(X)$;
b) *if* $n \in \{2, 3\}$, *then* $\mathcal{E}_n(X) = \Lambda_n(X)$;
c) *if* $n \geq 4$, *then* $\mathcal{E}_n(X) = \Lambda_n(X) - F_{n-1}(X)$.

Proof. To show a) let $A \in \mathcal{E}_n(X)$. By Theorem 4.2, $A \cap \mathrm{cl}_X(T(X)) = \emptyset$. Since $X \in \mathcal{D}$, this implies that $A \cap (R(X) \cup E_a(X)) = \emptyset$. Thus $A \in \Lambda_n(X)$, so a) holds. Assertion b) follows from a) and the proof of [7, Lemma 5.1]. To show c) assume that $n \geq 4$. Take $A \in \mathcal{E}_n(X)$. By a), $A \in \Lambda_n(X)$. Let $\mathcal{U}$ be a neighborhood of A in $F_n(X)$ which is an n-cell. Then $\mathcal{U}$ can be embedded in $\mathbb{R}^n$ so, by Theorem 3.4, $A \notin F_{n-1}(X)$. This shows that $\mathcal{E}_n(X) \subset \Lambda_n(X) - F_{n-1}(X)$. The other inclusion holds by Theorem 4.3. □

Theorem 4.6. *Let $X \in \mathcal{D}$ and $A \in F_n(X)$. If $A \cap E_a(X) = \emptyset$, then there exists a basis $\mathfrak{B}$ of open neighborhoods of A in $F_n(X)$ such that for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is nonempty and has a finite number of components.*

Proof. Since $A \cap E_a(X) = \emptyset$, we have $A \in F_n(X) - EA_n(X)$. Thus, by Theorem 3.1, there is a tree T in X such that $A \subset \mathrm{int}_X(T)$ and $T \cap E_a(X) = \emptyset$. Let $A = \{x_1, x_2, \ldots, x_m\}$ and consider that A has exactly m points. Let $\varepsilon > 0$. Choose a finite collection $U_1, U_2, \ldots, U_m$ of pairwise disjoint open connected subsets of X with the following properties:

1) $x_i \in U_i \subset \mathrm{int}_X(T) \cap B_X(x_i, \varepsilon)$, for each $i \in \{1, 2, \ldots, m\}$;
2) $U_i - \{x_i\} \subset O(X)$, for each $i \in \{1, 2, \ldots, m\}$;

Let $\mathcal{V}_\varepsilon = \langle U_1, U_2, \ldots, U_m \rangle_n$. By 1) we have $\mathcal{V}_\varepsilon \subset B_{F_n(X)}(A, \varepsilon)$ and, by Theorem 4.4, $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X) \neq \emptyset$. Given $i \in \{1, 2, \ldots, m\}$, since $X \in \mathcal{D}$, the order of x_i in X is finite. From this and 2), the set

$$U_i \cap [O(X) \cup (E(X) - E_a(X))]$$

has a finite number of components. Let $\left\{J_1^i, J_2^i, \ldots, J_{l_i}^i\right\}$ be the set of components of $U_i \cap [O(X) \cup (E(X) - E_a(X))]$. By Theorem 3.2 the components of $\mathcal{V}_\varepsilon \cap \Lambda_n(X)$ are the nonempty sets of the form:

$$\left\langle J_{s_1}^{r_1}, J_{s_2}^{r_2}, \ldots, J_{s_k}^{r_k} \right\rangle_n \tag{4.1}$$

where $\{r_1, r_2, \ldots, r_k\} = \{1, 2, \ldots, m\}$, the sets $J_{s_1}^{r_1}, J_{s_2}^{r_2}, \ldots, J_{s_k}^{r_k}$ are pairwise different and $s_t \in \{1, 2, \ldots, l_{r_t}\}$, for every $t \in \{1, 2, \ldots, k\}$. Since we have a finite number of elements of the form $J_{s_t}^{r_i}$, the number of nonempty sets of the form (4.1) is finite.

If $n \in \{2, 3\}$ then, by part b) of Theorem 4.5, the nonempty sets of the form (4.1) are the components of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. Assume then that $n \geq 4$. Then, by part c) of Theorem 4.5, $\mathcal{E}_n(X) = \Lambda_n(X) - F_{n-1}(X)$. Given a component $\mathcal{C} = \left\langle J^{r_1}_{s_1}, J^{r_2}_{s_2}, \ldots, J^{r_k}_{s_k} \right\rangle_n$ of $\mathcal{V}_\varepsilon \cap \Lambda_n(X)$ and $(q_1, q_2, \ldots, q_k) \in \mathbb{N}^k$ such that $q_1 + q_2 + \cdots + q_k = n$ let:

$$\mathcal{C}(q_1, q_2, \ldots, q_k) = \left\{ C \in \mathcal{C} \colon \left| C \cap J^{r_t}_{s_t} \right| = q_t \text{ for each } t \in \{1, 2, \ldots, k\} \right\}.$$

Note that $\mathcal{C}(q_1, q_2, \ldots, q_k) \subset \mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. It is not difficult to see that $\mathcal{C}(q_1, q_2, \ldots, q_k)$ is homeomorphic to

$$\left(F_{q_1}(J^{r_1}_{s_1}) - F_{q_1 - 1}(J^{r_1}_{s_1}) \right) \times \cdots \times \left(F_{q_k}\left(J^{r_k}_{s_k}\right) - F_{q_k - 1}\left(J^{r_k}_{s_k}\right) \right),$$

where we agree that $F_0(R) = \emptyset$ for each continuum R. Since the sets

$$F_{q_1}\left(J^{r_1}_{s_1}\right) - F_{q_1 - 1}\left(J^{r_1}_{s_1}\right), \ldots, F_{q_k}\left(J^{r_k}_{s_k}\right) - F_{q_k - 1}\left(J^{r_k}_{s_k}\right)$$

are connected, $\mathcal{C}(q_1, q_2, \ldots, q_k)$ is a connected subset of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. Moreover

$$\mathcal{C} \cap \mathcal{E}_n(X) = \bigcup \left\{ \mathcal{C}(q_1, \ldots, q_k) \colon (q_1, \ldots, q_k) \in \mathbb{N}^k \text{ and } q_1 + \cdots + q_k = n \right\}.$$

This implies that $\mathcal{C} \cap \mathcal{E}_n(X)$ has a finite number of components. Since each component of $\mathcal{C} \cap \mathcal{E}_n(X)$ is a component of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$ and $\mathcal{V}_\varepsilon \cap \Lambda_n(X)$ has a finite number of components, the set $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$ has a finite number of components as well.

To finish the proof note that $\mathfrak{B} = \{\mathcal{V}_\varepsilon \colon \varepsilon > 0\}$ is a basis of open neighborhoods of A in $F_n(X)$. ☐

In [4] and [20] it is proved that locally connected continua admit a convex metric d. This means that every two points $x, y \in X$ can be joined by an arc J in X, in such a way that J is isometric to the closed interval $[0, d(x, y)]$.

Theorem 4.7. *Let $X \in \mathcal{D}$ and $A \in F_n(X)$. Assume that $A \cap E(X) \neq \emptyset$. Then there exists a basis $\mathfrak{B}$ of open neighborhoods of A in $F_n(X)$ such that, for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} - \{A\}$ is contractible. Moreover if $A \cap E_a(X) \neq \emptyset$ we can choose $\mathfrak{B}$ with the additional property that, for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ has infinitely many components.*

Proof. Let d be a convex metric on X. Assume that $|A| = m$. Let $A = \{x_1, x_2, \ldots, x_m\}$ and assume that $x_1 \in E(X)$. Let $\varepsilon > 0$. Choose a finite collection $U_1, U_2, \ldots, U_m$ of pairwise disjoint open connected subsets of X such that $x_i \in U_i \subset B_X(x_i, \varepsilon)$, for each $i \in \{1, 2, \ldots, m\}$. Let $\mathcal{V}_\varepsilon = \langle U_1, U_2, \ldots, U_m \rangle_n$. Clearly $A \in \mathcal{V}_\varepsilon \subset B_{F_n(X)}(A, \varepsilon)$. Assume that $\operatorname{diam}(U_i) < 1$, for each $i \in \{1, 2, \ldots, m\}$. Fix $B = \{b_1, b_2, \ldots, b_m\}$ so that $b_1 \in U_1 - \{x_1\}$ and $b_i \in U_i$ for each $i \in \{2, 3, \ldots, m\}$. Note that $B \in \mathcal{V}_\varepsilon - \{A\}$. Given $i \in \{1, 2, \ldots, m\}$ and $(x, t) \in U_i \times I$, by [21, Theorem 8.26], $[x, b_i] \subset U_i$. We also have that $[x, b_i]$ is isometric to the closed interval $[0, d(x, b_i)]$. Hence if $d(x, b_i) \geq t$ there is a unique point $y_x \in [x, b_i]$ such that $d(x, y_x) = t$. We can then define a function $g_i \colon U_i \times I \to U_i$ by:

$$g_i(x, t) = \begin{cases} b_i, & \text{if } d(x, b_i) \leq t; \\ y_x, & \text{if } d(x, b_i) \geq t. \end{cases}$$

It is not difficult to prove that g_i is a continuous function. Note that $g_i(x,0) = x$ and $g_i(x,1) = b_i$, for all $x \in U_i$. If $x \in U_1 - \{x_1\}$ then $[x, b_1] \subset U_1 - \{x_1\}$ so, by the definition of g_1, we have $g_1(x,t) \in U_1 - \{x_1\}$ for every $t \in I$.

Define $G\colon (\mathcal{V}_\varepsilon - \{A\}) \times I \to \mathcal{V}_\varepsilon - \{A\}$ so that if $(D,t) \in (\mathcal{V}_\varepsilon - \{A\}) \times I$, then:

$$G(D,t) = \bigcup_{i=1}^{m} g_i\left((D \cap U_i) \times \{t\}\right).$$

It is not difficult to see that G is well defined and continuous. Since $G(D,0) = \bigcup_{i=1}^{m}(D \cap U_i) = D$ and $G(D,1) = B$, for each $D \in \mathcal{V}_\varepsilon - \{A\}$, the set $\mathcal{V}_\varepsilon - \{A\}$ is contractible.

Let us assume now that $x_1 \in E_a(X)$. Since $X \in \mathcal{D}$, each element of $E_a(X)$ is the limit of a sequence of distinct ramification points of X, all in the same arc. We also have, since $X \in \mathcal{D}$, that $R(X)$ is discrete ([3, Corollary 3.6]). Then we can find a sequence $(r_k)_k$ in $R(X) \cap U_1$ such that:

1) $(r_k)_k$ converges to x_1;
2) (r_{k+1}, r_k) is an internal arc in X, for every $k \in \mathbb{N}$;
3) $r_{k+1} \in (r_{k+2}, r_k) \subset U_1$, for each $k \in \mathbb{N}$.

Given $i \in \{2,3,\ldots,m\}$ fix an arc I_i in $\mathrm{cl}_X(U_i)$ which is either external or internal in $\mathrm{cl}_X(U_i)$. Let $J_i = \mathrm{int}_{U_i}(I_i \cap U_i)$. By Theorems 3.2 and 4.5, for every $k \in \mathbb{N}$, the set:

$$\mathcal{W}_k = \langle J_2, J_3, \ldots, J_m, (r_{k+1}, r_k), (r_{k+2}, r_{k+1}), \ldots, (r_{k+n-m+1}, r_{k+n-m})\rangle_n$$

is a component of $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$. Since $\mathcal{W}_k \cap \mathcal{W}_l = \emptyset$, if $k \neq l$, the set $\mathcal{V}_\varepsilon \cap \mathcal{E}_n(X)$ has infinitely many components.

To finish the proof, note that $\mathfrak{B} = \{\mathcal{V}_\varepsilon \colon \varepsilon > 0\}$ is a basis of open neighborhoods of A in $F_n(X)$ as required. □

Theorem 4.8. *Let X be a locally connected continuum and Z be a nondegenerate subcontinuum of X such that $\mathrm{cl}_X(T(X) \cap Z) = Z$. Assume that there is a point $p \in Z$ such that $p \in \mathrm{int}_X(Z)$. Then there exists a basis $\mathfrak{B}$ of open neighborhoods of $\{p\}$ in $F_n(X)$ such that, for each $\mathcal{V} \in \mathfrak{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is empty.*

Proof. Take $\varepsilon > 0$ such that $B_X(p,\varepsilon) \subset \mathrm{int}_X(Z)$. Let

$$\mathfrak{B} = \left\{B_{F_n(X)}(\{p\},\delta) \colon \delta < \varepsilon\right\},$$

$\mathcal{V} \in \mathfrak{B}$ and $A \in \mathcal{V} \cap F_n(Z)$. Since $\mathrm{cl}_X(T(X) \cap Z) = Z$, we have $A \cap \mathrm{cl}_X(T(X)) \neq \emptyset$. Thus, by Theorem 4.2, $A \notin \mathcal{E}_n(X)$. This implies that $\mathcal{V} \cap \mathcal{E}_n(X) = \emptyset$. □

Let X be a continuum and A be an arc in X with end points p and q. We say that A is a *free arc of* X if $A - \{p,q\}$ is an open subset of X.

Theorem 4.9. *Let X be a locally connected continuum and $n \in \mathbb{N}$ such that $\mathcal{E}_n(X)$ is dense in $F_n(X)$. Then, for each nonempty open subset U of X, there is a free arc of X contained in U.*

Proof. Assume, to the contrary, that U contains no free arcs. Let V be a nonempty open connected subset of X such that $\mathrm{cl}_X(V) \subset U$. Define $Z = \mathrm{cl}_X(V)$. We prove that $Z = \mathrm{cl}_X(T(X) \cap Z)$. Let $y \in Z$ and W be an open subset of X such that $y \in W$. Let $p \in W \cap V$ and A be an arc such that $p \in A \subset V \cap W$. Since U has no free arcs and open subsets of X are locally arcwise connected (see [21, Definition 8.24 and Theorem 8.25]) it can be shown that there is $a \in A \cap T(X) \cap W$. Thus $W \cap T(X) \cap Z \neq \emptyset$. This shows that $Z \subset \mathrm{cl}_X(T(X) \cap Z)$ and, since the other inclusion also holds, we have $\mathrm{cl}_X(T(X) \cap Z) = Z$. Since the interior of Z is nonempty, by Theorem 4.8, there is an open set $\mathcal{V}$ in $F_n(X)$ such that $\mathcal{V} \cap \mathcal{E}_n(X) = \emptyset$. This contradicts the fact that $\mathcal{E}_n(X)$ is dense in $F_n(X)$. Therefore U contains a free arc. □

5. The Main Theorem

We start this section by showing the following result, which is a positive answer to [15, Question 2]. In its proof we will use the fact that a continuum Z is locally connected if and only if $F_n(Z)$ is locally connected ([9, Theorem 6.3]), and also that if Z is a one-dimensional continuum, then $\dim(F_n(Z)) = n$. This follows from [6, Theorem 3] and [10, Proof of Lemma 3.1].

Theorem 5.1. *Let X be a dendrite and $n \in \mathbb{N}$. If Y is a continuum such that $F_n(X) \approx F_n(Y)$, then Y is a dendrite.*

Proof. Since X is locally connected, Y is also locally connected. By [8, Theorem 1.1(19)], $\dim(X) = 1$. Thus $\dim(F_n(Y)) = \dim(F_n(X)) = n$. Assume that $\dim(Y) > 1$. Then there exist $q \in Y$ and a compact neighborhood B of q such that $\dim(B) \geq 2$. Such B can be chosen so that there is a finite collection $A_1, A_2, \ldots, A_{n-1}$ of pairwise disjoint arcs in Y such that $B \cap A_i = \emptyset$, for all $i \in \{1, 2, \ldots, n-1\}$. Then $\mathcal{B} = \langle B, A_1, A_2, \ldots, A_{n-1}\rangle_n$ is a subset of $F_n(Y)$ which is homeomorphic to $B \times A_1 \times A_2 \times \cdots \times A_{n-1}$. Since B is compact and $\dim(A_i) = 1$ for every $i \in \{1, 2, \ldots, n-1\}$, by [14, Remark, p. 34], $\dim(\mathcal{B}) = \dim(B \times A_1 \times A_2 \times \cdots \times A_{n-1}) = \dim(B) + n - 1 \geq n + 1$. Hence $\dim(F_n(Y)) \geq n + 1$. Since this is a contradiction, $\dim(Y) = 1$.

Assume that Y contains a simple closed curve S^1. Since $\dim(Y) = 1$, by [22, 18.8, p. 104], there is a retraction $r\colon Y \to S^1$. Consider the function $R\colon F_n(Y) \to F_n\left(S^1\right)$ defined, for $A \in F_n(Y)$, by $R(A) = r(A)$. It is not difficult to see that R is a well defined retraction. Since X is contractible ([8, Theorem 1.2(21)]), $F_n(X)$ is contractible. Thus $F_n\left(S^1\right)$ is a retract of the contractible space $F_n(Y)$, so $F_n\left(S^1\right)$ is contractible as well ([5, Theorem 13.2]). However, in [24] it is shown that there is no $n \in \mathbb{N}$ so that $F_n(S^1)$ is contractible. This contradiction shows that Y does not contain a simple closed curve. We conclude that Y is a dendrite. □

Let X be a dendrite and K be a subcontinuum of X. Define $r\colon X \to K$ as follows: $r(x) = x$ if $x \in K$ and, otherwise, $r(x)$ is the unique point in K such that $r(x)$ is a point of every arc in X from x to any point of K (see [21, Lemma 10.24]). In [21, Lemma 10.25] it is shown that r is a retraction.

Such function is called the *first point map for K*. We use this function in the proof of the following result.

Theorem 5.2. *Let $X \in \mathcal{D}$ and $n \in \mathbb{N}$. If Y is a continuum such that $F_n(X) \approx F_n(Y)$, then $Y \in \mathcal{D}$.*

Proof. Since $X \approx F_1(X)$, the result is true for $n = 1$, so we consider that $n \geq 2$. By Theorem 5.1, Y is a dendrite. Let us assume that the metric d for Y is convex. If $p, q \in \mathbb{R}^2$, we denote by $[p, q]$ the straight line segment in $\mathbb{R}^2$ joining p and q. We consider that $(p, q) = [p, q] - \{p, q\}$.

Assume, to the contrary, that $Y \notin \mathcal{D}$. Then, by [3, Theorem 3.3], Y contains either a copy of

$$F_\omega = [(-1,0),(1,0)] \cup \left(\bigcup_{m=1}^{\infty} \left[(0,0), \left(\frac{1}{m}, \frac{1}{m^2} \right) \right] \right)$$

or of

$$W = [(-1,0),(1,0)] \cup \left(\bigcup_{m=1}^{\infty} \left[\left(-\frac{1}{m}, 0 \right), \left(-\frac{1}{m}, \frac{1}{m} \right) \right] \right).$$

To simplify notation let us assume that either $F_\omega \subset Y$ or $W \subset Y$. Note that $(0,0) \in \mathrm{cl}_Y(E(Y)) - E(Y)$. Let $x_1 = (0,0)$. Since $O(Y)$ is dense in Y, we can take $n-1$ points $x_2, x_3 \ldots, x_n$ in $O(Y) \cap ((0,0),(1,0))$. Let

$$B = \{x_1, x_2, \ldots, x_n\}.$$

Let $h\colon F_n(X) \to F_n(Y)$ be a homeomorphism. We will proceed as follows: after proving Claim 1, we consider the cases $h^{-1}(B) \cap E_a(X) = \emptyset$ and $h^{-1}(B) \cap E_a(X) \neq \emptyset$. In both situations we will find a contradiction. Thus the assumption $Y \notin \mathcal{D}$ is not correct and, in this way, the proof of the theorem will be complete.

By Theorems 4.1 and 4.4, $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$ and $\mathcal{E}_n(Y)$ is dense in $F_n(Y)$.

Take $\delta > 0$ such that $B_Y(x_i, \delta) \cap B_Y(x_j, \delta) = \emptyset$ for each $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.

Claim 1. For each open neighborhood $\mathcal{V}$ of B in $F_n(Y)$ with $\mathcal{V} \subset B_{F_n(Y)}(B, \delta)$, the set $\mathcal{V} \cap \mathcal{E}_n(Y)$ has infinitely many components.

To show Claim 1, let $\mathcal{V}$ be an open neighborhood of B in $F_n(Y)$ such that $\mathcal{V} \subset B_{F_n(Y)}(B, \delta)$. Let $0 < \varepsilon < \delta$ be such that the sets $B_Y(x_1, \varepsilon)$, $B_Y(x_2, \varepsilon), \ldots, B_Y(x_n, \varepsilon)$ are pairwise disjoint and

$$\langle B_Y(x_1, \varepsilon), B_Y(x_2, \varepsilon), \ldots, B_Y(x_n, \varepsilon) \rangle_n \subset \mathcal{V}.$$

Since $x_1 \in B_Y(x_1, \varepsilon)$ and either $F_\omega \subset Y$ or $W \subset Y$, there exists $N \in \mathbb{N}$ such that either

$$\bigcup_{m=N}^{\infty} \left[(0,0), \left(\frac{1}{m}, \frac{1}{m^2} \right) \right] \subset B_Y(x_1, \varepsilon) \tag{5.1}$$

or

$$\bigcup_{m=N}^{\infty}\left[\left(-\frac{1}{m},0\right),\left(-\frac{1}{m},\frac{1}{m}\right)\right]\subset B_Y\left(x_1,\varepsilon\right). \tag{5.2}$$

Also, since Y is a dendrite, $x_1\in \mathrm{cl}_Y(E(Y))-E(Y)$ and, according the case, the sequences $\left(\left(\frac{1}{m},\frac{1}{m^2}\right)\right)_m$ or $\left(\left(-\frac{1}{m},0\right)\right)_m$ and $\left(\left(-\frac{1}{m},\frac{1}{m}\right)\right)_m$ converge to x_1, we can take N so that, for every $m\geq N$, if (5.1) holds then the component of $B_Y\left(x_1,\varepsilon\right)-\{x_1\}$ that contains $\left(\frac{1}{m},\frac{1}{m^2}\right)$ coincides with the component of $Y-\{x_1\}$ that contains $\left(\frac{1}{m},\frac{1}{m^2}\right)$; and if (5.2) holds, then the component of $B_Y\left(x_1,\varepsilon\right)-\left\{\left(-\frac{1}{m},0\right)\right\}$ that contains $\left(-\frac{1}{m},\frac{1}{m}\right)$ coincides with the component of $Y-\left\{\left(-\frac{1}{m},0\right)\right\}$ that contains $\left(-\frac{1}{m},\frac{1}{m}\right)$.

Given $m\geq N$ we define Z_m as follows: if (5.1) holds, then Z_m is the component of $Y-\{x_1\}$ that contains $\left(\frac{1}{m},\frac{1}{m^2}\right)$ and, if (5.2) holds, then Z_m is the component of $Y-\left\{\left(-\frac{1}{m},0\right)\right\}$ that contains $\left(-\frac{1}{m},\frac{1}{m}\right)$. Since each Z_m is open in Y and $\mathcal{E}_n(Y)$ is dense in $F_n(Y)$, by Theorem 4.9, there is a free arc A_m of Y contained in Z_m. Note that $\{\mathrm{int}_Y(A_m): m\geq N\}$ is a sequence of pairwise disjoint open connected subsets of $B_Y\left(x_1,\varepsilon\right)$.

Given $i\in\{2,3,\ldots,n\}$, since $B_Y\left(x_i,\varepsilon\right)$ is open in Y and $\mathcal{E}_n(Y)$ is dense in $F_n(Y)$, by Theorem 4.9, there is a free arc J_i of Y contained in $B_Y\left(x_i,\varepsilon\right)$. Note that $\mathrm{int}_Y(J_2),\ldots,\mathrm{int}_Y(J_n)$ is a finite sequence of pairwise disjoint open connected subsets of Y.

For $m\geq N$ define

$$\mathcal{A}_m=\langle \mathrm{int}_Y(A_m),\mathrm{int}_Y(J_2),\ldots,\mathrm{int}_Y(J_n)\rangle_n.$$

Since $\mathrm{int}_Y(A_m),\mathrm{int}_Y(J_2),\ldots,\mathrm{int}_Y(J_n)$ are open connected subsets of Y, by Theorem 2.1, $\mathcal{A}_m$ is an open connected subset of $F_n(Y)$. Since $\mathrm{int}_Y(A_m)\cap \mathrm{int}_Y(A_k)=\emptyset$ if $m\neq k$, we have $\mathcal{A}_m\cap\mathcal{A}_k=\emptyset$. Given $C\in\mathcal{A}_m$, by Theorem 2.1, the set $\langle A_m,J_2,\ldots,J_n\rangle_n$ is an n-cell in $F_n(Y)$ that contains C in its interior. Thus $C\in\mathcal{E}_n(Y)$, so $\mathcal{A}_m\subset\mathcal{E}_n(Y)$. Moreover, we have

$$\mathcal{A}_m\subset\langle B_Y\left(x_1,\varepsilon\right),B_Y\left(x_2,\varepsilon\right),\ldots,B_Y\left(x_n,\varepsilon\right)\rangle_n\subset\mathcal{V},$$

so $\mathcal{A}_m\subset\mathcal{V}\cap\mathcal{E}_n(Y)$.

Let $\mathcal{B}_m$ be the component of $\mathcal{V}\cap\mathcal{E}_n(Y)$ that contains $\mathcal{A}_m$. We claim that $\mathcal{B}_m\cap\mathcal{B}_k=\emptyset$ for different $m,k\geq N$. Assume, to the contrary, that $\mathcal{B}_m=\mathcal{B}_k$. Let $D_1\in\mathcal{A}_m$ and $D_2\in\mathcal{A}_k$. Since $F_n(Y)$ is locally connected and $\mathcal{B}_m$ is a component of the open subset $\mathcal{V}\cap\mathcal{E}_n(Y)$ of $F_n(Y)$, the set $\mathcal{B}_m$ is arcwise connected. Then there is an arc $\alpha\colon[0,1]\to\mathcal{B}_m$ such that $\alpha(0)=D_1$ and $\alpha(1)=D_2$. Let

$$K=\bigcup\{\alpha(t)\colon t\in[0,1]\}.$$

Given $j\in\{1,2,\ldots,n\}$, let $K_j=K\cap B_Y(x_j,\delta)$. Since $\alpha([0,1])$ is connected in $F_n(Y)$, the subset K of Y has at most n components ([9, Lemma 6.1]). Since $D_1,D_2\subset K$, the sets $B_Y(x_1,\delta),B_Y(x_2,\delta),\ldots,B_Y(x_n,\delta)$ are pairwise disjoint and $D_i\cap B_Y(x_j,\delta)\neq\emptyset$, for each $i\in\{1,2\}$ and every $j\in\{1,2,\ldots,n\}$, it follows that $K_1,K_2,\ldots,K_n$ are the components of K.

Note that $K_1 \cap A_m \neq \emptyset$ and $K_1 \cap A_k \neq \emptyset$ so, if (5.1) holds, then $x_1 \in K_1$ and, if (5.2) holds, then $\left(-\frac{1}{m}, 0\right) \in K_1$. This implies that $K \cap R(Y) \neq \emptyset$, so one element of $\mathcal{B}_m$ contains a ramification point of Y. This contradicts Theorem 4.2. Hence $\mathcal{B}_m \cap \mathcal{B}_k = \emptyset$.

Therefore $\mathcal{V} \cap \mathcal{E}_n(Y)$ has infinitely many components. This completes the proof of Claim 1.

Let us assume that $h^{-1}(B) \cap E_a(X) = \emptyset$. Then, by Theorem 4.6, there exists a basis $\mathfrak{B}_X$ of open neighborhoods of $h^{-1}(B)$ in $F_n(X)$ such that, for each $\mathcal{U} \in \mathfrak{B}_X$, the set $\mathcal{U} \cap \mathcal{E}_n(X)$ is nonempty and has a finite number of components. Let $\mathfrak{B}_Y = \{h(\mathcal{U}) \colon \mathcal{U} \in \mathfrak{B}_X\}$. Then $\mathfrak{B}_Y$ is a basis of open neighborhoods of B in $F_n(Y)$ such that, for each $\mathcal{V} \in \mathfrak{B}_Y$, the set $\mathcal{V} \cap \mathcal{E}_n(Y)$ is nonempty and has a finite number of components. Let $\mathcal{V} \in \mathfrak{B}_Y$ such that $\mathcal{V} \subset B_{F_n(Y)}(B, \delta)$. By Claim 1) the set $\mathcal{V} \cap \mathcal{E}_n(Y)$ has infinitely many components. This is a contradiction.

Let us assume now that $h^{-1}(B) \cap E_a(X) \neq \emptyset$. Then, by Theorem 4.7, there is a basis $\mathfrak{B}$ of open neighborhoods of $h^{-1}(B)$ in $F_n(X)$ such that, for each $\mathcal{U} \in \mathfrak{B}$, the set $\mathcal{U} - \{h^{-1}(B)\}$ is contractible. Let $\mathfrak{C} = \{h(\mathcal{U}) \colon \mathcal{U} \in \mathfrak{B}\}$. Then $\mathfrak{C}$ is a basis of open neighborhoods of B in $F_n(Y)$ such that, for each $\mathcal{V} \in \mathfrak{C}$, the set $\mathcal{V} - \{B\}$ is contractible.

Let $A = [(-1,0),(1,0)]$. Note that A is an arc in Y such that $x_1 = (0,0) \in ((-1,0),(1,0))$.

Claim 2. There is a retraction $r \colon Y \to A$ such that $r^{-1}(x_i) = \{x_i\}$, for each $i \in \{1, 2, \ldots, n\}$.

To show Claim 2, let $r_1 \colon Y \to A$ be the first point map for A. By [21, Lemma 10.25], r_1 is a retraction. Given $i \in \{2, 3, \ldots, n\}$, since $x_i \in O(X)$, we have $r_1^{-1}(x_i) = \{x_i\}$. If $x_1 \in O(Y)$, then $r_1^{-1}(x_1) = \{x_1\}$ and r_1 has the required properties. If $x_1 \notin O(Y)$, then $r_1^{-1}(x_1) = \{y \in Y \colon [y, x_1] \cap A = \{x_1\}\}$. Let $A_0 = [(-1,0),(0,0)]$. Given $y \in r_1^{-1}(x_1)$, if $d(x_1, y) \leq d(x_1, (-1,0))$, there is a unique $z_y \in A_0$ such that $d(x_1, z_y) = d(x_1, y)$. Then we can define a function $r_2 \colon r_1^{-1}(x_1) \to A_0$ so that:

$$r_2(y) = \begin{cases} z_y, & \text{if } d(x_1, y) \leq d(x_1, (-1,0)); \\ (-1,0), & \text{if } d(x_1, y) \geq d(x_1, (-1,0)). \end{cases}$$

It is not difficult to see that r_2 is a well defined continuous function such that $r_2^{-1}(x_1) = x_1$. Now define $r \colon Y \to A$ so that, if $y \in Y$, then:

$$r(y) = \begin{cases} r_1(y), & \text{if } y \notin r_1^{-1}(x_1); \\ r_2(y), & \text{if } y \in r_1^{-1}(x_1). \end{cases}$$

Then r is a retraction such that $r^{-1}(y) = \{x_i\}$, for each $i \in \{1, 2, ..., n\}$. This proves Claim 2.

Let $r \colon Y \to A$ as in Claim 2. Define $R \colon F_n(Y) \to F_n(A)$, at $D \in F_n(Y)$, by $R(D) = r(D)$. Then R is a retraction such that $R^{-1}(B) = B$. For each $\varepsilon > 0$ with $\varepsilon < \delta$, let $U_i^{\varepsilon} = B_Y(x_i, \varepsilon)$ for $i \in \{1, 2, \ldots, n\}$ and $\mathcal{U}^{\varepsilon} = \langle U_1^{\varepsilon}, U_2^{\varepsilon}, \ldots, U_n^{\varepsilon}\rangle_n$.

Claim 3. For each $\varepsilon > 0$ with $\varepsilon < \delta$, the set $R(\mathcal{U}^{\varepsilon})$ is a connected open subset of $F_n(A)$ homeomorphic to the Euclidean space $\mathbb{R}^n$.

Considering the sets $U_1^{\varepsilon}, U_2^{\varepsilon}, \ldots, U_n^{\varepsilon}$ are pairwise disjoint, it is not difficult to prove that $R(\mathcal{U}^{\varepsilon}) = \langle r(U_1^{\varepsilon}), r(U_2^{\varepsilon}), \ldots, r(U_n^{\varepsilon})\rangle_n$. Since the metric for Y is convex, by the definition of r, $r(U_1^{\varepsilon}), r(U_2^{\varepsilon}), \ldots, r(U_n^{\varepsilon})$ are open connected subsets of A. Thus $R(\mathcal{U}^{\varepsilon})$ is an open connected subset of $F_n(A)$. Moreover, since A is an arc, by Theorem 2.1, $R(\mathcal{U}^{\varepsilon})$ is homeomorphic to $\mathbb{R}^n$. This proves Claim 3.

Now we are ready to show the final argument. Fix $\gamma > 0$ with $\gamma < \delta$ and take $\mathcal{V} \in \mathfrak{C}$ such that $B \in \mathcal{V} \subset \mathcal{U}^{\gamma}$. Now let $\sigma > 0$ such that $B \in \mathcal{U}^{\sigma} \subset \mathcal{V} \subset \mathcal{U}^{\gamma}$. Since R is a retraction, $\mathcal{V} - \{B\}$ is contractible and $R(\mathcal{V} - \{B\}) = R(\mathcal{V}) - \{B\}$, then the set $R(\mathcal{V}) - \{B\}$ is contractible.

Since $B \in \mathcal{U}^{\sigma} \subset \mathcal{V} \subset \mathcal{U}^{\gamma}$, by the definition of R, $B \in R(\mathcal{U}^{\sigma}) \subset R(\mathcal{V}) \subset R(\mathcal{U}^{\gamma})$. Thus, by Claim 3, $R(\mathcal{U}^{\sigma})$ is an open neighborhood of B homeomorphic to $\mathbb{R}^n$ and contained in the set $R(\mathcal{V})$. Then there exists an n-cell G such that $B \in G \subset R(\mathcal{V})$ and $B \notin \partial G$, where ∂G is the manifold boundary of G. By Claim 3, $R(\mathcal{U}^{\gamma})$ is also homeomorphic to $\mathbb{R}^n$ so there is a retraction $S\colon R(\mathcal{U}^{\gamma}) - \{B\} \to \partial G$. Then $S_{|R(\mathcal{V})-\{B\}}\colon R(\mathcal{V}) - \{B\} \to \partial G$ is also a retraction. Since $R(\mathcal{V}) - \{B\}$ is contractible, the set ∂G is contractible. This is a contradiction to [14, p. 37] that came from the assumption that $h^{-1}(B) \cap E_a(X) \neq \emptyset$.

Since both cases $h^{-1}(B) \cap E_a(X) = \emptyset$ and $h^{-1}(B) \cap E_a(X) \neq \emptyset$ produced a contradiction, the assumtion that $Y \notin \mathcal{D}$ is not correct. Therefore $Y \in \mathcal{D}$. ☐

Acknowledgement. We would like to thank professor Alejandro Illanes for his suggestions for this paper. We would also like to thank the referee for his/her fruitful comments for the improvement of this paper.

References

[1] G. Acosta, *Continua with unique hyperspace*, Lecture Notes in Pure and Applied Mathematics, 230, 33–49, Marcel Dekker, Inc. 2002.

[2] G. Acosta and D. Herrera-Carrasco, *Dendrites without unique hyperspace*, to appear in Houston J. Math.

[3] D. Arévalo, W. J. Charatonik, P. Pellicer-Covarruvias and L. Simón, *Dendrites with a closed set of end points*, Topology Appl. 115 (2001), 1–17.

[4] R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc., 55 (1949), 1101-1110.

[5] K. Borsuk, *Theory of Retracts*, Monografie Matematyczne, vol. 44, Polish Scientific Publishers, Warszawa, Poland, 1967.

[6] K. Borsuk and S. Ulam, *On symmetric products of topological spaces*, Bull. Amer. Math. Soc., 37 (1931), 875–882.

[7] E. Castañeda and A. Illanes, *Finite graphs have unique symmetric products*, Topology Appl. 153 (2006), 1434–1450.

[8] J. J. Charatonik, *Dendrites*, Aportaciones Mat. Comun. 22 (1998), 227–253.

[9] J.J. Charatonik and A. Illanes, *Local Connectedness in Hyperspaces*, Rocky Mountain J. Math. 36 (2006), 811–856.

[10] D. Curtis and N. T. Nhu, *Hyperspaces of finite subsets which are homeomorphic to* $\aleph_0$*-dimensional linear metric spaces*, Topology Appl. 19 (1985), 251–260.

[11] D. Herrera-Carrasco, *Dendrites with unique hyperspace*, Houston J. Math. 33 (3) (2007), 795–805.

[12] D. Herrera-Carrasco and F. Macías-Romero, *Dendrites with unique* n*-fold hyperspace*, preprint.

[13] D. Herrera-Carrasco, M. de J. López and F. Macías-Romero, *Dendrites with unique simmetric products*, preprint.

[14] W. Hurewicz and H. Wallman, *Dimension Teory*, Princeton University Press, 1948.

[15] A. Illanes, *Dendrites with unique hyperspace* $F_2(X)$, JP Jour, Geometry & Topology 2 (1) (2002), 75–96.

[16] A. Illanes, *Dendrites with unique hyperspace* $C_2(X)$, *II*, preprint.

[17] A. Illanes and S. B. Nadler Jr., *Hyperspaces: Fundamentals and Recent Advances*, vol. 216 of *Pure and Applied Mathematics*, Marcel Dekker, Inc., New York and Basel, 1999.

[18] K. Kuratowski, *Topology*, Vol. 2, Academic Press and PWN, New York, London and Warszawa, 1968.

[19] J. M. Martinez-Montejano, *Non-confluence of the natural map of products onto symmetric products*, Lecture Notes in Pure and Applied Mathematics, 230, 229–236, Marcel Dekker, Inc. 2002.

[20] E. E. Moise, *Grille decomposition and convexification theorems for compact locally connected continua*, Bull. Amer. Math. Soc. 55 (1949), 1111-1121.

[21] S. B. Nadler, Jr., *Continuum Theory: An Introduction*, Monographs and Textbooks in Pure and Applied Math., Vol. 158, Marcel Dekker, New York, Basel and Hong Kong, 1992.

[22] S. B. Nadler, Jr., *Dimension Theory: An Introduction with Exercises*, Aportaciones Matemáticas: Textos [Mathematical Contributions: Texts], 18. Sociedad Matemática Mexicana, México, 2002.

[23] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Providence, RI, 1942, reprinted with corrections 1971.

[24] W. Wu, *Note sur les produits essentiels symétriques des espaces topologiques, I*, (French) Comptes Rendus des Séances de l'Académie des Sciences, 16 (1947), 1139–1141.

(G. Acosta) INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIUDAD UNIVERSITARIA, MÉXICO D.F., 04510, MÉXICO
E-mail address: `gacosta@matem.unam.mx`

(R. Hernández-Gutiérrez) INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIUDAD UNIVERSITARIA, MÉXICO D.F., 04510, MÉXICO
E-mail address: `rod@matem.unam.mx`

(V. Martínez-de-la-Vega) INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIUDAD UNIVERSITARIA, MÉXICO D.F., 04510, MÉXICO
E-mail address: `vmvm@matem.unam.mx`