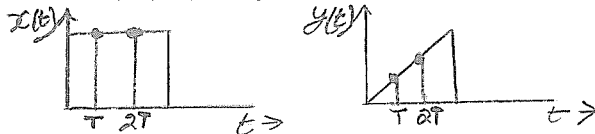


## Comparison of Two Signals Using Cross-Correlation

[Note: Please read Sections of 2.5.1, 2.6 of Lathi's text for more details about this topic.]

### Introduction

Cross-correlation is a very useful tool to quantify or measure the degree of similarity between two signals. Consider, for instance, the following two continuous-time signals,  $x(t)$  and  $y(t)$ , as well as their discretized forms,  $x(nT)$  and  $y(nT)$ , where  $T$  denotes the sampling interval.



A commonly used measure of similarity between  $x(t)$  and  $y(t)$  is called their cross-correlation, which is given by:

$$\text{Corr}[x(t), y(t)] = \int_{-\infty}^{\infty} x(t)y(t)dt \quad (1a)$$

( which is sometimes also denoted by  $\langle x(t), y(t) \rangle$  )

In discretized form, we can write equation (1) as:

$$\text{Corr}[x(n), y(n)] = \sum_{n=-\infty}^{\infty} x(n)y(n) \quad (1b)$$

where  $x(n)$  and  $y(n)$  are commonly used alternative notations for  $x(nT)$  and  $y(nT)$ , respectively.

In the example in Fig. 1,  $T_s$  is chosen such that we get just two samples of  $x(t)$  and  $y(t)$ . Thus, we obtain

$$\text{Corr}[x(n), y(n)] = \sum_{n=1}^2 x(n)y(n) \quad (2)$$

### Interpretation in terms of vectors

Suppose

$$X = \begin{bmatrix} x(1) \\ x(2) \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y(1) \\ y(2) \end{bmatrix}$$

denote two vectors in a rectangular coordinate space (with axes representing the values at  $T_s$  and  $2T_s$ , respectively). Then we can think of equation (2) as the dot product of the vectors,  $X$  and  $Y$ , i.e.,

$$\text{Corr}[x(n), y(n)] = X \cdot Y \quad (3)$$

Also, we know that

$$X \cdot Y = |X||Y|\cos(\theta), \quad (4a)$$

where  $\theta$  denotes the angle between the vectors, X and Y and

$$|X| = \sqrt{[x(1)^2 + x(2)^2]} \text{ and } |Y| = \sqrt{[y(1)^2 + y(2)^2]} \quad (4b)$$

The quantity,

$$\cos(\theta) = (X \cdot Y) / (|X||Y|) \quad (5)$$

is taken to be a measure of similarity between the vectors, X and Y, because

$\theta = 0 \Rightarrow \cos(\theta) = 1$ , i.e, X and Y are similar in the sense,  $Y = \alpha X$ , where  $\alpha > 0$  is a constant;

$\theta = 180^\circ \Rightarrow \cos(\theta) = -1$ , i.e, X and Y are similar in opposite sense,  $Y = -\alpha X$ ;

$\theta = 90^\circ \Rightarrow \cos(\theta) = 0$ , i.e, X and Y are orthogonal to each other.

Notice that the quantity in the numerator of equation (5), i.e.,  $X \cdot Y$ , which is also denoted as the inner product,  $X^T Y$ , of two column vectors X and Y, equals the actual (raw) cross-correlation between X and Y. It is denoted by:

$$r_{xy} = X^T Y \quad (6a)$$

Also, the quantity in the denominator of equation (5), i.e.,  $|X||Y|$ , just serves as a normalization factor. Thus, equation (5) can also be rewritten as:

$$\rho_{xy} = (X^T Y) / (|X||Y|) \quad (6)$$

which is known as the normalized cross-correlation, or correlation coefficient of X and Y. In view of what was said above,

$\rho_{xy} = 1 \Rightarrow$  maximum positive correlation between X and Y (i.e., they are similar),

$\rho_{xy} = -1 \Rightarrow$  maximum negative correlation between X and Y (similar in negative sense),

$\rho_{xy} = 0 \Rightarrow$  X and Y are uncorrelated (or, dissimilar).

### Generalization of the above idea

Instead of just two samples, we can collect N samples of  $x(t)$  and  $y(t)$  and let

$$X = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ \vdots \\ x(N) \end{bmatrix} \quad Y = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ \vdots \\ y(N) \end{bmatrix}$$

Then we can define

$$\rho_{xy} = (X^T Y) / (|X||Y|) = \frac{\sum_{n=1}^N x(n)y(n)}{\sqrt{\left[ \sum_{n=1}^N x^2(n) \sum_{n=1}^N y^2(n) \right]}} \quad (7)$$

Notice that the above equation can also be written as:

$$\rho_{xy} = (X^T Y) / (|X||Y|) = \frac{\sum_{n=1}^N x(n)y(n)}{\sqrt{E_x E_y}} \quad (8)$$

where  $E_x$  and  $E_y$  denote the energies of  $x(n)$  and  $y(n)$ , respectively.

Equation (8) is used to compute the normalized cross-correlation of two discretized signals,  $x(n)$  and  $y(n)$ . Notice that if the sampling interval  $T$  is small, and  $N \rightarrow \infty$ , we get the cross-correlation of the corresponding analog signals,  $x(t)$  and  $y(t)$ , i.e.,

$$\rho_{xy} = (X^T Y) / (|X||Y|) = \frac{\int_{-\infty}^{\infty} x(t)y(t) dt}{\sqrt{\left[ \int_{-\infty}^{\infty} x^2(t) dt \int_{-\infty}^{\infty} y^2(t) dt \right]}} \quad (9)$$

### Applications of cross-correlation

Cross-correlation is used to solve a variety of signal processing and communication problems, such as:

1. Target range estimation in a Radar
2. Detection of 1s and 0s in a Correlation Receiver
3. Detection of abnormalities in data
4. Identification and detection of known types of signals (or signatures) in presence of noise, etc.

Two simple examples of applications 1 and 2 are explored in Lab #3.