First, we begin by noting that in an elastic collision, momentum and kinetic energy is conserved. Start off with a statement of the conservation of momentum:

$$m_1 v_{1b} + m_2 v_{2b} = m_1 v_{1a} + m_2 v_{2a} \quad \text{rewriting . . .} \tag{1}$$
$$m_1(v_{1b} - v_{2b}) = -m_2(v_{2b} - v_{2a}) \tag{2}$$

Here I use a subscript b to signify before the collision and a for after the collision. We also give a statement for an elastic collision ⇒ kinetic energy is conserved. If we add up each object's kinetic energy before and after we get a little something like this (note: I dropped the one halves in front of each term because they cancel):

$$m_1 v_{1b}^2 + m_2 v_{2b}^2 = m_1 v_{1a}^2 + m_2 v_{2a}^2 \quad \text{rewriting . . .} \tag{3}$$
$$m_1(v_{1b}^2 - v_{2b}^2) = -m_2(v_{2b}^2 - v_{2a}^2) \tag{4}$$

If I use the fact that $a^2 - b^2 = (a - b)(a + b)$ then I can rewrite Eq. 4 as this:

$$m_1(v_{1b} - v_{2b})(v_{1b} + v_{2b})q = -m_2(v_{2b} - v_{2a})v_{2b} + v_{2a}) \tag{5}$$

If I divide Eq. 5 by Eq. 2 then I will get something like this:

$$\frac{m_1(v_{1b} - v_{2b})(v_{1b} + v_{2b})}{m_1(v_{1b} - v_{2b})} = \frac{-m_2(v_{2b} - v_{2a})(v_{2b} + v_{2a})}{-m_2(v_{2b} - v_{2a})} \quad \text{cancelling terms} \tag{6}$$
$$v_{1b} + v_{1a} = v_{2b} + v_{2a} \quad \text{solve this for } v_{1a} \tag{7}$$
$$v_{1a} = v_{2b} + v_{2a} - v_{1b} \quad \text{and solve this for } v_{2a} \tag{8}$$
$$v_{2a} = v_{1b} + v_{2a} - v_{2b} \tag{9}$$

If we plug Eq. 8 into Eq. 1, we find

$$m_1 v_{1b} + m_2 v_{2b} = m_1 v_{1a} + m_2 v_{2a} \tag{10}$$
$$m_1 v_{1b} + m_2 v_{2b} = m_1(v_{2b} + v_{2a} - v_{1b}) + m_2 v_{2a} \quad \text{collecting terms} \tag{11}$$
$$m_1 v_{1b} + m_1 v_{1b} + m_2 v_{2b} - m_1 v_{2b} = v_{2a} m_1 + v_{2a} m_2 \tag{12}$$
$$2m_1 v_{1b} + (m_2 - m_1)v_{2b} = v_{2a}(m_1 + m_2) \quad \text{Finally we get} \tag{13}$$
$$v_{2a} = \frac{2m_1}{(m_1 + m_2)} v_{1b} + \frac{(m_2 - m_1)}{(m_1 + m_2)} v_{2b} \tag{14}$$

Now if we plug Eq. 9 into Eq. 1 we will get:

$$m_1 v_{1b} + m_2 v_{2b} = m_1 v_{1a} + m_2 v_{2a} \tag{15}$$
$$m_1 v_{1b} + m_2 v_{2b} = m_1 v_{1a} + m_2(v_{1b} + v_{2a} - v_{2b}) \quad \text{collecting like terms} \tag{16}$$
$$m_1 v_{1b} - m_2(v_{1b} + m_2 v_{2b} + m_2 v_{2b} = m_1 v_{1a} + m_2 v_{1a} \quad \text{Finally we have} \tag{17}$$
$$v_{1a} = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_{1b} + \frac{2m_2}{(m_1 + m_2)} v_{2b} \tag{18}$$

Finally for the example in class, $m_1 = 2$ kg, $m_2 = 6$ kg, $v_{1b} = 6m/sec$ and $v_{2b} = -4m/sec$. Plug in these values (dropping units) into Eq. 18 to get

$$v_{1a} = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_{1b} + \frac{2m_2}{(m_1 + m_2)} v_{2b} \tag{19}$$
$$= \frac{(2-6)}{(2+6)} 6 + \frac{2*6}{(2+6)}(-4) = \frac{-4}{8} * 6 + \frac{2*6}{(2+6)}(-4) = -3 + -6 = -9m/sec \tag{20}$$

and

$$v_{2a} = \frac{2m_1}{(m_1 + m_2)} v_{1b} + \frac{(m_2 - m_1)}{(m_1 + m_2)} v_{2b} \tag{21}$$
$$= \frac{2*2}{(2+6)} 6 + \frac{(6-2)}{(2+6)}(-4) = 3 + (-2) = 1m/sec \tag{22}$$