

# LINEAR QUADRATIC OPTIMIZATION FOR MODELS WITH RATIONAL EXPECTATIONS

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Abstract. In this paper we present a method for using rational expectations in a linear-quadratic optimization framework. Following the approach put forward by Sims, we solve the model through a QZ decomposition, which is generally easier to implement than the more widely used method of Blanchard and Kahn.

## 1. Introduction

Starting in the late 1970's there was much criticism of the use of control theory in economics. In this view one of the major drawbacks was that it could not deal with rational expectations. Subsequently a number of generic methods to solve models with rational expectations (RE) were developed. For instance, Fair and Taylor (1983) use an iterative method for solving RE models and, in the tradition of Theil (1968), Fisher, Holly and Hughes Hallett (1986) use a method based on stacking the model variables. A hybrid method based on the saddlepoint property is presented in Anderson and Moore (1985).

The Blanchard and Kahn method (1980) is another well known method for solving linear models with RE in discrete time. By decomposing the model into stable and unstable parts, the unstable part can be solved backward in time and the stable part forward in time. Although the BK approach is theoretically a powerful method, for practical implementation it has some serious drawbacks. First, not all linear RE models can be put into first order linear form as required by the Blanchard and Kahn method, and second, the method is based on the *Jordan canonical form*. The procedure to put the model into the Jordan canonical is not widely available in software libraries and is known to be numerically unstable, see Moler (1994).

In recent work, Amman, Kendrick and Achath (1995), Amman (1996), we presented a procedure that introduces RE in a linear-quadratic (LQ) control framework based on the Blanchard and Kahn method. Due to the limitations of the Blanchard and Kahn approach and the fact that we had to rely on the diagonalization of the transition matrix, this work could only deal with a limited set of models. Recently, Sims (1996) proposed a different method for solving linear models with RE allowing for a broader range of models. This method is not based on the Jordan canonical form, but uses the more

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widely available *QZ* form that is based on generalized eigenvalues and which is more numerically stable. In this paper we will follow the paper of Sims and incorporate his approach into the LQ framework.

## 2. Problem statement and solution

Following Kendrick (1981), the standard single-agent linear-quadratic optimization problem is written as:

Find the set of admissible instruments  $U = \{u_0, u_1, \dots, u_{T-1}\}$  that minimizes the welfare loss function

$$J_T = \beta^T L_T(x_T) + \sum_{t=0}^{T-1} \beta^t L_t(x_t, u_t) \quad (1)$$

with

$$\begin{aligned} L_T &= \frac{1}{2}(x_T - \bar{x}_T)' W_T (x_T - \bar{x}) \\ L_t &= \frac{1}{2}(x_t - \bar{x}_t)' W_t (x_t - \bar{x}_t) + \\ &\quad \frac{1}{2}(u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t) + (x_t - \bar{x}_t)' F_t (u_t - \bar{u}_t) \end{aligned}$$

subject to the model

$$x_{t+1} = A_t x_t + B_t u_t + C_t z_t \quad (2)$$

The vector  $x_t \in \mathfrak{R}^n$  is the state of the economy at time  $t$  and the vector  $u_t \in \mathfrak{R}^m$  contains the policy instruments. The initial state of the economy  $x_0$  is known,  $\bar{x}_t$  and  $\bar{u}_t$  are target values.  $W_t$ ,  $R_t$  and  $F_t$  are penalty matrices of conformable size,  $\beta$  being a discount factor.

The above model is straightforward to solve and there are a number of packages available. However, a serious drawback for economics is that equation (2) does not allow for rational expectations. One way of allowing RE to enter the model is to augment equation (2) in the following fashion

$$x_{t+1} = A_t x_t + B_t u_t + C_t z_t + \sum_{j=1}^k D_{j,t} E_t x_{t+j} + \epsilon_t \quad (2a)$$

where the matrix  $D_{j,t}$  is a parameter matrix,  $E_t x_{t+1}$  is the expected state for time  $t + j$  at time  $t$ ,  $k$  the maximum lead in the expectations formation and  $\epsilon_t$  is a white noise vector. In order to compute the admissible set of instruments we have to eliminate the rational expectations from the model. In a previous paper Amman, Kendrick and Achath (1995), we used the Blanchard and Kahn method to solve the RE in the model. However, as mentioned above, the Blanchard and Kahn method has some features that impede practical implementation. Sims (1996) proposes a different route by applying a

*generalized eigenvalue* approach. In order to apply Sims' method we first put equation (2a) in the form (see Appendix A)<sup>1</sup>

$$\Gamma_{0,t}\tilde{x}_{t+1} = \Gamma_{1,t}\tilde{x}_t + \Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_{4,t}\epsilon_t \quad (3)$$

where

$$\Gamma_{0,t} = \begin{bmatrix} I - D_{1,t} & -D_{2,t} & \dots & -D_{k-1,t} & -D_{k,t} \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & & I & 0 \end{bmatrix}$$

$$\Gamma_{1,t} = \begin{bmatrix} A_t & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & I \end{bmatrix} \quad \Gamma_{2,t} = \begin{bmatrix} B_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_{3,t} = \begin{bmatrix} C_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_{4,t} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and the augmented state vector

$$\tilde{x}_t = \begin{bmatrix} x_t \\ Ex_{t+1} \\ Ex_{t+2} \\ \vdots \\ Ex_{t+k-1} \end{bmatrix} \quad (4)$$

Taking the generalized eigenvalues of equation (3) allows us to decompose the system matrices  $\Gamma_{0,t}$  and  $\Gamma_{1,t}$  in the following manner, *viz.* Coleman and Van Loan (1988) or Moler and Stewart (1973) ,

$$\Lambda_t = Q_t \Gamma_{0,t} Z_t$$

$${}_{t-1} = Q_t \Gamma_{1,t} Z_t$$

with  $Z_t' Z_t = I$  and  $Q_t' Q_t = I$ . The matrices  $\Lambda_t$  and  ${}_{t-1}$  are upper triangular matrices and the generalized eigen values are  $\forall i \omega_{i,i}/\lambda_{i,i}$ . If we use the transformation  $w_t = Z_t' \tilde{x}_t$  and  $w_{t+1} = Z_t' \tilde{x}_{t+1}$  we can write equation (3) as (see Appendix B)

$$\Lambda_t w_{t+1} = {}_{t-1} w_t + Q_t \Gamma_{2,t} u_t + Q_t \Gamma_{3,t} z_t + Q_t \Gamma_{4,t} \epsilon_t \quad (5)$$

It is possible to reorder the matrices  $Z_t$  and  $Q_t$  in such a fashion that the diagonal elements of the matrices  $\Lambda_t$  and  ${}_{t-1}$  contain the generalized eigenvalues in ascending order. In that case we can write equation (4) as follows

<sup>1</sup>Note that in contrast to Sims (1996) the variable  $z_t$  contains exogenous variables and not random variables. Hence, the matrix  $\Pi$  in Sims' paper is set to zero.

$$\begin{bmatrix} \Lambda_{11,t} & \Lambda_{12,t} \\ 0 & \Lambda_{22,t} \end{bmatrix} \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} 11,t & 12,t \\ 0 & 22,t \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_{2,t} u_t + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_{3,t} z_t + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_{4,t} \epsilon_t \quad (6)$$

where the unstable eigen values are in lower right corner, that is the matrices  $\Lambda_{22,t}$  and  ${}_{22,t}$ . By forward propagation and taking expectations (see Appendix C), it is possible to derive  $w_{2,t}$  as a function of future instruments and exogenous variables, Sims (1996, page 5)

$$\gamma_t = w_{2,t} = - \sum_{j=0}^{\infty} \tilde{M}_{t,j} \begin{matrix} -1 \\ {}_{22,t+j} \end{matrix} Q_{2,t+j} (\Gamma_{2,t+j} u_{t+j} + \Gamma_{3,t+j} z_{t+j}) \quad (7)$$

The matrix  $\tilde{M}_{t,j}$  is defined as

$$\tilde{M}_{t,j} = \prod_{i=0}^{j-1} M_{t+i} \quad \text{for } j > 0$$

and

$$\tilde{M}_{t,j} = I \quad \text{for } j = 0$$

with

$$M_t = \begin{matrix} -1 \\ {}_{22,t} \end{matrix} \Lambda_{22,t}$$

Given the fact that  ${}_{22,t}$  contains the eigenvalues outside the unit circle, we have applied the follow condition in deriving equation (7)

$$\lim_{j \rightarrow \infty} \tilde{M}_{t,j} = 0$$

In contrast to Sims,  $M_t$  is not time invariant since we explicitly want to allow for time dependent matrices in the model<sup>2</sup>. Reinserting equation (7) into equation (6) gives us

$$\tilde{\Lambda}_t w_{t+1} = \tilde{~}_t w_t + \tilde{\Gamma}_{2,t} u_t + \tilde{\Gamma}_{3,t} z_t + \tilde{\Gamma}_{4,t} \epsilon_t + \tilde{\gamma}_t \quad (8)$$

with

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_{11,t} & \Lambda_{12,t} \\ 0 & I \end{bmatrix} \quad \tilde{~} = \begin{bmatrix} 11,t & 12,t \\ 0 & 0 \end{bmatrix} \quad \tilde{\Gamma}_{2,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_{2,t}$$

$$\tilde{\Gamma}_{3,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_{3,t} \quad \tilde{\Gamma}_{4,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_{4,t} \quad \tilde{\gamma}_t = \begin{bmatrix} 0 \\ \gamma_t \end{bmatrix}$$

Knowing that  $\tilde{x}_t = Z_t w_t$  and  $\tilde{x}_{t+1} = Z_t w_{t+1}$  we can write equation (8) as

<sup>2</sup>As Lucas (1976) pointed out the parameters in the model might change due to policy announcements. Hence we would like our method to be able to deal with changing parameters in the system matrices.

$$\tilde{x}_{t+1} = \tilde{A}_t \tilde{x}_t + \tilde{B}_t u_t + \tilde{C}_t \tilde{z}_t + \tilde{\epsilon}_t \quad (9)$$

with

$$\tilde{A}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{Z}'_t \quad \tilde{B}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{2,t} \quad \tilde{C}_t = [Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{3,t} \quad Z_t \tilde{\Lambda}_t^{-1}] \quad (10)$$

and

$$\tilde{\Lambda}_t^{-1} = \begin{bmatrix} \Lambda_{11,t}^{-1} & -\Lambda_{11,t}^{-1} \Lambda_{12,t} \\ 0 & I \end{bmatrix} \quad \tilde{z}_t = \begin{bmatrix} z_t \\ \tilde{\gamma}_t \end{bmatrix} \quad \tilde{\epsilon}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{4,t} \epsilon_t \quad (11)$$

In non-pathological cases, Stewart (1973), the matrix  $\Lambda_{11,t}$  will be nonsingular. With equation (9) we have transformed equation (2a) into the form required by equation (2) enabling us to set up an iterative scheme using the LQ framework in equations (1)-(2). Knowing that  $\gamma_t$  depends on the future the basic algorithm works like this <sup>3</sup>

Step 0. Set the iteration count  $\nu = 0$  and set the instruments  $u_t^\nu$ ,  $t = \{0, 1, \dots, T+s-1\}$ , set  $\tilde{x}_0$  (see below).

Step 1. Compute  $\gamma_t^\nu$ ,  $t = \{0, 1, \dots, T+s-1\}$ , by using

$$\gamma_t^\nu = - \sum_{j=0}^{\infty} \tilde{M}_{t,j}^{-1} Q_{2,t+j} (\Gamma_{2,t+j} u_{t+j}^\nu + \Gamma_{3,t+j} z_{t+j})$$

Step 2. Apply the LQ framework in equations (1)-(2) to compute a new set of optimal instruments  $u_t^{\nu+1}$  using the equation below in place of equation (2)

$$\tilde{x}_{t+1}^{\nu+1} = \tilde{A}_t \tilde{x}_t^{\nu+1} + \tilde{B}_t u_t^{\nu+1} + \tilde{C}_t \tilde{z}_t^\nu$$

Step 3. Set  $\nu = \nu + 1$  and go to Step 1 until convergence has been reached.

There is still one remaining issue we have to take care of. In order to apply the algorithm for  $\nu = 0$  we need to have an initial value of the augmented state vector, which is

$$\tilde{x}_0 = \begin{bmatrix} x_0 \\ Ex_1 \\ Ex_2 \\ \vdots \\ Ex_{k-1} \end{bmatrix} \quad (12)$$

However, at the beginning of the first iteration step the expectational variables are not known, so we have to give the expectational part of  $\tilde{x}_0^0$  an arbitrary value. Once we have completed the first iterations we can compute the states to reinitialize  $\tilde{x}_0^1$ . By iterating on the expectations the above algorithm produces an expectation consistent solution of

<sup>3</sup>The Matlab implementation of this algorithm can be obtained through the corresponding author.

the model formulated in equations (1) and (2a)<sup>4</sup>. In the next section we will apply this algorithm using a simple macro model.

### 3. An example

Consider a simple macro model with output,  $x_t$ , consumption,  $c_t$ , investment,  $i_t$ , government expenditures,  $g_t$ , and taxes  $\tau_t$ . The problem can then be stated as:

Find for the model

$$x_{t+1} = c_{t+1} + i_{t+1} + g_{t+1} \quad (13)$$

$$c_{t+1} = 0.8(x_t - \tau_t) + 200 \quad (14)$$

$$i_{t+1} = 0.2E_t x_{t+2} + 100 + \epsilon_t \quad (15)$$

$$g_{t+1} = u_t \quad (16)$$

$$\tau_{t+1} = 0.25x_{t+1} \quad (17)$$

with  $x_0 = 1500$ , a set of admissible control  $U = \{u_0, u_1, \dots, u_9\}$  to minimize the welfare loss function

$$J_T = \frac{1}{2}(x_{10} - 1600)^2 + \frac{1}{2} \sum_{t=0}^9 \{(x_t - 1600)^2 + g_t^2\} \quad (18)$$

If we reduce the above model to one equation for output we get

$$x_{t+1} = 0.6x_t + u_t + 0.2E_t x_{t+2} + 300 + \epsilon_t \quad (19)$$

which leads to the augmented system

$$\begin{bmatrix} 1 & -0.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 300 \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix} \quad (20)$$

where  $z_t = 1 \forall t$ . Applying the QZ factorization, Coleman and van Loan (1988), to compute the generalized eigen values of the model gives us the time invariant solution

$$\Lambda = \begin{bmatrix} 1.0822 & -0.9136 \\ 0 & 0.1848 \end{bmatrix} = \begin{bmatrix} 0.7546 & 0.3979 \\ 0 & 0.7951 \end{bmatrix} \quad (21)$$

$$Z = \begin{bmatrix} 0.8202 & -0.5719 \\ 0.5719 & 0.8203 \end{bmatrix} \quad Q = \begin{bmatrix} 0.6523 & 0.7580 \\ -0.7580 & 0.6523 \end{bmatrix} \quad (22)$$

so the eigenvalues are  $\{0.7546/1.0822, 0.7952/0.1848\} = \{0.6972, 4.3028\}$  and apparently the ordering of the system is such that the unstable root 4.3028 is in the lower right corner of  $\Lambda$  and . The other components are

$$\tilde{A} = \begin{bmatrix} 0.2966 & 0.5745 \\ 0.2068 & 0.4006 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0.4944 \\ 0.3447 \end{bmatrix} \quad (23)$$

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<sup>4</sup>It is beyond the scope of this paper to address fully the issue of time-consistency. The above algorithm produces a time consistent, but not necessarily, fully optimal solution of the model. See Kydland and Prescott (1977) or Hall and Henry (1988).

$$\tilde{C} = \begin{bmatrix} 148.3243 & 0.7580 & 0.1205 \\ 103.4153 & 0.5285 & 1.3031 \end{bmatrix} \quad (24)$$

$$\tilde{W} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = [1] \quad \tilde{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (25)$$

$$\tilde{x}_0 = \begin{bmatrix} 1500 \\ 1500 \end{bmatrix} \quad \tilde{\tilde{x}}_0 = \begin{bmatrix} 1600 \\ 0 \end{bmatrix} \quad \bar{u} = [0] \quad (26)$$

Note that we have set  $E_0 x_1^0 = 1500$ . For this simple model we can compute the steady state of the system being  $x_\infty = 1589.08$  and  $u_\infty = 17.81$ , see Amman and Kendrick (1998). Hence,  $U^0 = \{17.81, \dots, 17.81\}$  is an educated guess for the instruments. The solution to the model is in Table 1.

**Table 1.**  
Solution of the LQ optimization model with RE

$t$	0	1	2	3	4	5	6	7	8	9	10
$x_t$	1500	1556	1576	1584	1587	1588	1589	1589	1587	1584	1578
$u_t$	40	26	21	19	18	18	18	17	16	11	

#### 4. Summary

In this paper we have presented a single agent Linear-Quadratic optimization model that allows for rational expectations. Based on Sims's paper we have used a generalized eigenvalue method for solving the variables that involve the unstable roots. By using an iterative scheme, the reduced model can be fitted into a standard Linear-Quadratic framework that allows us to derive the optimal policy instruments for the model with rational expectations.

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**Appendix A**  
**Derivation of Equation 3**

Begin with equation (2a), i.e.

$$x_{t+1} = A_t x_t + B_t u_t + C_t z_t + \sum_{j=1}^k D_{jt} E_t x_{t+j} + \epsilon_t \quad (2a)$$

For simplicity it is useful to show the derivation for the case  $k = 3$ . Also, move the expectation terms to the left hand side of the equation. In this case equation (2a) becomes

$$x_{t+1} - D_{1t} E_t x_{t+1} - D_{2t} E_t x_{t+2} - D_{3t} E_t x_{t+3} = A_t x_t + \eta_t \quad (A-1)$$

where

$$\eta_t = B_t u_t + C_t z_t + \epsilon_t \quad (A-2)$$

Next use the assumption that

$$x_{t+1} = E_t x_{t+1} \quad (A-3)$$

in order to rewrite equation (A-1) as

$$(I - D_{1t})x_{t+1} - D_{2t}E_t x_{t+2} - D_{3t}E_t x_{t+3} = A_t x_t + \eta_t \quad (A-4)$$

Then augment the state vector by adding two additional vectors of state variables. Call the augmented state vector  $\tilde{x}$ -tilde

$$\tilde{x}_t = \begin{bmatrix} x_t \\ s_t \\ v_t \end{bmatrix} = \begin{bmatrix} x_t \\ E_t x_{t+1} \\ E_t x_{t+2} \end{bmatrix} \quad (A-5)$$

Also using equation (A-5) we obtain

$$\tilde{x}_{t+1} = \begin{bmatrix} x_{t+1} \\ s_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \\ E_t x_{t+3} \end{bmatrix} \quad (A-6)$$

Then using elements of equation (A-6) we can rewrite equation (A-4) as

$$(I - D_{1t})x_{t+1} - D_{2t}s_{t+1} - D_{3t}v_{t+1} = A_t x_t + \eta_t \quad (A-7)$$

Also from Eqs. (A-5) and (A-6) and using equation (A-3) we have

$$x_{t+1} = E_t x_{t+1} = s_t \quad (A-8)$$

and

$$s_{t+1} = E_t x_{t+2} = v_t \quad (A-9)$$

Then Eqs. (A-7) thru (A-9) can be rewritten as

$$\begin{bmatrix} I - D_{1t} & -D_{2t} & -D_{3t} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ s_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_t \\ s_t \\ v_t \end{bmatrix} + \begin{bmatrix} \eta_t \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-10})$$

or using Eqs. (A-5) and (A-6) as

$$\begin{bmatrix} I - D_{1t} & -D_{2t} & -D_{3t} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \\ E_t x_{t+3} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_t \\ E_t x_{t+1} \\ E_t x_{t+2} \end{bmatrix} + \begin{bmatrix} \eta_t \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-11})$$

Recall from equation (A-5) that the augmented state vector is

$$\tilde{x}_t = \begin{bmatrix} x_t \\ E_t x_{t+1} \\ E_t x_{t+2} \end{bmatrix} \quad (\text{A-12})$$

Then use equation (A-2) and equation (A-12) to rewrite equation (A-11) as

$$\Gamma_{0t} \tilde{x}_{t+1} = \Gamma_{1t} \tilde{x}_t + \Gamma_{2t} u_t + \Gamma_{3t} z_t + \Gamma_{4t} \epsilon_t \quad (\text{A-13})$$

where

$$\Gamma_{0t} = \begin{bmatrix} I - D_{1t} & -D_{2t} & -D_{3t} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \quad \Gamma_{1t} = \begin{bmatrix} A_t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\Gamma_{2t} = \begin{bmatrix} B_t \\ 0 \\ 0 \end{bmatrix} \quad \Gamma_{3t} = \begin{bmatrix} C_t \\ 0 \\ 0 \end{bmatrix} \quad \Gamma_{4t} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

equation (A-13) is the desired result since it has the same structure as equation (3) in the body of the paper.

## Appendix B Derivation of Equation 5

Begin with equation (3), i.e.

$$\Gamma_{0,t} \tilde{x}_{t+1} = \Gamma_{1,t} \tilde{x}_t + \Gamma_{2,t} u_t + \Gamma_{3,t} z_t + \Gamma_{4,t} \epsilon_t \quad (\text{B-1})$$

From the equation just above equation (5)

$$\Lambda_t = Q_t \Gamma_{0,t} Z_t \quad (\text{B-2})$$

$${}_t = Q_t \Gamma_{1,t} Z_t \quad (\text{B-3})$$

Premultiply equations (B-2) and (B-3) by  $Q_t'$  and post multiply them by  $Z_t'$  keeping in mind that  $Z_t' Z_t = I$  and  $Q_t' Q_t = I$  yields

$$Q_t' \Lambda_t Z_t' = \Gamma_{0,t} \quad (\text{B-4})$$

$$Q_t' {}_t Z_t' = \Gamma_{1,t} \quad (\text{B-5})$$

Then substitute equations (B-4) and (B-5) into equation (B-1) to obtain

$$Q_t' \Lambda_t Z_t' \tilde{x}_{t+1} = Q_t' {}_t Z_t' \tilde{x}_t + \Gamma_{2,t} u_t + \Gamma_{3,t} z_t + \Gamma_{4,t} \epsilon_t \quad (\text{B-6})$$

Premultiplication of equation (B-6) by  $Q_t$  and use of the definitions

$$w_{t+1} = Z_t' \tilde{x}_{t+1} \quad (\text{B-7})$$

$$w_t = Z_t' \tilde{x}_t \quad (\text{B-8})$$

yields

$$\Lambda_t w_{t+1} = {}_t w_t + Q_t \Gamma_{2,t} u_t + Q_t \Gamma_{3,t} z_t + Q_t \Gamma_{4,t} \epsilon_t \quad (\text{B-9})$$

which is equation (5) in the body of the paper.

### Appendix C Derivation of Equation 7

Begin with the bottom half of equation (6), i.e.

$$\Lambda_{22,t} w_{2,t+1} = {}_{22,t} w_{2,t} + Q_{2t} \Gamma_{2,t} u_t + Q_{2t} \Gamma_{3,t} z_t + Q_{2t} \Gamma_{4,t} \epsilon_t \quad (\text{C-1})$$

For the sake of simplification define

$$\xi_t = Q_{2t} (\Gamma_{2,t} u_t + \Gamma_{3,t} z_t + \Gamma_{4,t} \epsilon_t) \quad (\text{C-2})$$

Then using equation (C-2) in equation (C-1) and rearranging terms we obtain

$${}_{22,t} w_{2,t} = \Lambda_{22,t} w_{2,t+1} - \xi_t \quad (\text{C-3})$$

or

$$w_{2,t} = \Lambda_{22,t}^{-1} w_{2,t+1} - \Lambda_{22,t}^{-1} \xi_t \quad (\text{C-4})$$

Then define

$$M_t = \frac{-1}{22,t} \Lambda_{22,t} \quad (\text{C-5})$$

and use equation (C-5) in equation (C-4) to obtain

$$w_{2,t} = M_t w_{2,t+1} - \frac{-1}{22,t} \xi_t \quad (\text{C-6})$$

Next solve equation (C-6). Begin this process by using equation (B-6) to obtain

$$w_{2,t+1} = M_{t+1} w_{2,t+2} - \frac{-1}{22,t+1} \xi_{t+1} \quad (\text{C-7})$$

Then substitute equation (C-7) into equation (C-6) to obtain

$$w_{2,t} = M_t (M_{t+1} w_{2,t+2} - \frac{-1}{22,t+1} \xi_{t+1}) - \frac{-1}{22,t} \xi_t \quad (\text{C-8})$$

or

$$w_{2,t} = M_t M_{t+1} w_{2,t+2} - M_t \frac{-1}{22,t+1} \xi_{t+1} - \frac{-1}{22,t} \xi_t \quad (\text{C-9})$$

Repeat the process above for  $w_{2,t+2}$ . First, from equation (C-7)

$$w_{2,t+2} = M_{t+2} w_{2,t+3} - \frac{-1}{22,t+2} \xi_{t+2} \quad (\text{C-10})$$

Then substitution of equation (C-10) into equation (C-9) yields

$$w_{2,t} = M_t M_{t+1} (M_{t+2} w_{2,t+3} - \frac{-1}{22,t+2} \xi_{t+2}) - M_t \frac{-1}{22,t+1} \xi_{t+1} - \frac{-1}{22,t} \xi_t \quad (\text{C-11})$$

or

$$w_{2,t} = M_t M_{t+1} M_{t+2} w_{2,t+3} - M_t M_{t+1} \frac{-1}{22,t+2} \xi_{t+2} - M_t \frac{-1}{22,t+1} \xi_{t+1} - \frac{-1}{22,t} \xi_t \quad (\text{C-12})$$

This process can be continued for  $s$  periods until one obtains

$$w_{2,t} = \left( \prod_{i=0}^s M_{t+i} \right) w_{2,t+s+1} - \sum_{j=1}^s \left( \prod_{i=0}^{j-1} M_{t+i} \right) \frac{-1}{22,t+j} \xi_{t+j} - \frac{-1}{22,t} \xi_t \quad (\text{C-13})$$

Under the condition that

$$\lim_{s \rightarrow \infty} \left( \prod_{i=0}^s M_{t+i} \right) = 0 \quad (\text{C-14})$$

equation (C-13) becomes in the limit as  $s \rightarrow \infty$

$$w_{2,t} = - \sum_{j=1}^{\infty} \left( \prod_{i=0}^{j-1} M_{t+i} \right) \frac{-1}{22,t+j} \xi_{t+j} - \frac{-1}{22,t} \xi_t \quad (\text{C-15})$$

Then define

$$\tilde{M}_{t,j} = \prod_{i=0}^{j-1} M_{t+i} \quad \text{for } j \geq 1 \quad (\text{C-16})$$

and

$$\tilde{M}_{t,j} = I \quad j = 0 \quad (\text{C-17})$$

and write equation (C-15) as

$$w_{2,t} = - \sum_{j=0}^{\infty} \tilde{M}_{t,j} \frac{-1}{22,t+j} \xi_{t+j} \quad (\text{C-18})$$

Substitution of equation (C-2) into equation (C-18) then yields after taking expectations

$$w_{2,t} = - \sum_{j=0}^{\infty} \tilde{M}_{t,j} \frac{-1}{22,t+j} Q_{2,t+j} (\Gamma_{2,t+j} u_{t+j} + \Gamma_{3,t+j} z_{t+j}) \quad (\text{C-19})$$

which is the same as equation (7) in the body of the paper.

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