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Stochastic Policy Design in a Learning Environment with Rational Expectations[1]

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Abstract. In this paper, we present a method for using rational expectations in a stochastic linear-quadratic optimization framework in which the unknown parameters are updated through a learning scheme. We use the QZ decomposition as suggested by Sims (Ref. 1) to solve the rational expectations part of the model. The parameter updating is done with the Kalman filter and the optimal control is calculated using the covariance matrix of the uncertain parameter.

Key Words. Macroeconomics, learning, rational expectations, stochastic optimization.

1. Introduction

There has been a recent revival of interest in learning under the title of bounded rationality [Marcet and Sargent (Ref. 2) and Sargent (Ref. 3)]. Earlier works on learning in macroeconomics include studies by Prescott (Ref. 4), MacRea (Ref. 5), Chow (Ref. 6), and Kendrick (Ref. 7). In a recent paper [Amman and Kendrick (Ref. 8)], we employed the Sims QZ decomposition approach to solve a rational expectations model in a deterministic optimal control context with parameter updating using the Ljung and Söderström (Ref. 9) self-tuning regulator. Here, we extend that work to the case where the parameters are unknown and treated as stochastic, and where the optimal instruments are computed while taking into account

[1]This paper is dedicated to David Luenberger for his many contributions to optimization methods and the application of these methods in economics and finance. David has not only led the way, but has provided the clearest kind of exposition as he went, thereby making easier the path that so many have followed.

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the variance and covariances of the parameter estimates. The learning of the parameters is done through the use of a Kalman filter.

2. Problem Statement

Following Kendrick (Ref. 7), the standard single-agent stochastic linear-quadratic (LQ) optimization problem is written as:

Find the set of admissible instruments $U = \{u_0, u_1, \ldots, u_{T-1}\}$ that minimizes the welfare loss function,

$$J_T = E\left\{\beta^T L_T(x_T) + \sum_{t=0}^{T-1} \beta^t L_t(x_t, u_t)\right\}, \tag{1a}$$

with

$$L_T = (1/2)(x_T - \tilde{x}_T)' W (x_T - \tilde{x}_T), \tag{1b}$$

$$L_t = (1/2)(x_t - \tilde{x}_t)' W (x_t - \tilde{x}_t) + (1/2)(u_t - \tilde{u}_t)' R (u_t - \tilde{u}_t) + (x_t - \tilde{x}_t)' F (u_t - \tilde{u}_t), \tag{1c}$$

subject to the model

$$x_{t+1} = A(\theta)x_t + B(\theta)u_t + c_t(\theta) + \epsilon_t. \tag{2a}$$

The vector $x_t \in \Re^n$ is the state of the economy at time t and the vector $u_t \in \Re^m$ contains the policy instruments. The initial state of the economy $x_0 \in \Re^n$ is known, $\tilde{x}_t \in \Re^n$ and $\tilde{u}_t \in \Re^m$ are target values; $W \in \Re^{n\times n}$, $R \in \Re^{m\times m}$ and $F \in \Re^{n\times m}$ are penalty matrices; $\epsilon_t \in \Re^n$ is a white noise vector with $\epsilon_t \sim N(0, \Sigma^{\epsilon\epsilon})$. We assume that $\Sigma^{\epsilon\epsilon} \in \Re^{n\times n}$ is known to the policy maker. Learning is introduced into the LQ framework by the unknown parameter vector $\theta \in \Re^p$ which is determined through a learning strategy.

The above model is straightforward to solve; see Kendrick (Ref. 7). However, a serious drawback for economics is that Eq. (2a) does not allow for RE. One way of allowing RE to enter the model is to augment Eq. (2a) in the following fashion:

$$x_{t+1} = A(\theta)x_t + B(\theta)u_t + c(\theta)_t + \sum_{j=1}^{k} D_j(\theta) E_t x_{t+j} + \epsilon_t, \tag{2b}$$

where $D_j(\theta)$ is a parameter matrix, $E_t x_{t+j}$ is the expected state for time $t+j$ as seen from time t, and k is the maximum lead in the expectations formation.[4]

[4]See also Amman (Ref. 10).

In order to compute the admissible set of instruments, we have to eliminate the rational expectations from the model. In an earlier paper, Amman and Kendrick (Ref. 8), we described how the control model with RE can be solved using the Sims approach (Ref. 1). In this case, we employ this method for the situations in which there is learning.

3. Solving Rational Expectations

For the sake of simplicity, let us assume that we have an estimate of the parameter vector $\hat{\theta}_{t|t}$. In our notations, this is the estimated value of θ at time t using the observations through time t. The covariance matrix of this parameter vector is defined as $\hat{\Sigma}^{\theta\theta}_{t|t}$. For the time being, we will treat $\hat{\theta}_{t|t}$ as being constant. In a later phase, we return to the issue of reestimating $\hat{\theta}_{t|t}$.

In the last decade, a number of generic methods to solve models with rational expectations were developed. For instance, Fair and Taylor (Ref. 11) use an iterative method for solving RE models; in the tradition of Theil (Ref. 12), Fisher, Holly, and Hughes Hallett (Ref. 13) use a method based on stacking the model variables. McCallum (Ref. 14) and Uhlig (Ref. 15) use the method of undetermined coefficients. A hybrid method based on the saddle-point property is presented in Anderson and Moore (Ref. 16).

Recently, Sims (Ref. 1) proposed a method based on the QZ decomposition. Following the work of Sims (Ref. 1), which is an extension of the work of Blanchard and Kahn (Ref. 17), we can rewrite the system Eq. (2a) in the following augmented form:

$$\Gamma_0 \tilde{x}_{t+1} = \Gamma_1 \tilde{x}_t + \Gamma_2 u_t + \Gamma_{3,t} + \Gamma_4 \epsilon_t, \tag{3}$$

where

$$\Gamma_0 = \begin{bmatrix} I - D_1(\hat{\theta}_{t|t}) & -D_2(\hat{\theta}_{t|t}) & \dots & -D_{k-1}(\hat{\theta}_{t|t}) & -D_k(\hat{\theta}_{t|t}) \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & & I & 0 \end{bmatrix}, \tag{4a}$$

$$\Gamma_1 = \begin{bmatrix} A(\hat{\theta}_{t|t}) & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & I \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} B(\hat{\theta}_{t|t}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4b}$$

$$\Gamma_{3,t} = \begin{bmatrix} c_t(\hat{\theta}_{t|t}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \Gamma_4 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4c}$$

and the augmented state vector is

$$\tilde{x}_t = \begin{bmatrix} x_t \\ Ex_{t+1} \\ Ex_{t+2} \\ \vdots \\ Ex_{t+k-1} \end{bmatrix}. \tag{5}$$

Taking the generalized eigenvalues of Eq. (3) allows us to decompose the system matrices Γ_0 and Γ_1 in the following manner [see Moler and Stewart (Ref. 18) or Coleman and Van Loan (Ref. 19)]:

$$\Lambda = Q\Gamma_0 Z, \qquad \Omega = Q\Gamma_1 Z,$$

with

$$Z'Z = I, \qquad Q'Q = I.$$

Here, Λ and Ω are upper triangular matrices and the generalized eigenvalues are $\omega_{i,i}/\lambda_{i,i}, \forall i$. If we use the transformation

$$w_t = Z'\tilde{x}_t,$$

we can write Eq. (3) as

$$\Lambda w_{t+1} = \Omega_1 w_t + Q\Gamma_2 u_t + Q\Gamma_{3,t} + Q\Gamma_4 \epsilon_t. \tag{6}$$

Given the triangular structure of Λ and Ω, we can partition (6) as follows:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix}$$

$$= \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Gamma_2 u_t + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Gamma_{3,t} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Gamma_4 \epsilon_t, \tag{7}$$

where the unstable eigenvalues are the ones in the lower-right corner, i.e., in the matrices Λ_{22} and Ω_{22}. By forward propagation and taking expectations, it is possible to derive $w_{2,t}$ as a function of future instruments and exogenous variables,

$$\gamma_t = w_{2,t} = -\sum_{j=0}^{\infty} M^{j-1}\Omega_{22}^{-1}Q_2(\Gamma_2 u_{t+j} + \Gamma_{3,t}), \tag{8}$$

with

$$M = \Omega_{22}^{-1}\Lambda_{22}.$$

Reinserting Eq. (8) into Eq. (6) gives us

$$\tilde{\Lambda}w_{t+1} = \tilde{\Omega}w_t + \tilde{\Gamma}_2 u_t + \tilde{\Gamma}_{3,t} + \tilde{\Gamma}_4\epsilon_t + \tilde{\gamma}_t, \tag{9}$$

with

$$\tilde{\Lambda} = \begin{bmatrix}\Lambda_{11} & \Lambda_{12}\\ 0 & I\end{bmatrix}, \qquad \tilde{\Omega} = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ 0 & 0\end{bmatrix}, \qquad \tilde{\Gamma}_2 = \begin{bmatrix}Q_1\\ 0\end{bmatrix}\Gamma_2,$$

$$\tilde{\Gamma}_{3,t} = \begin{bmatrix}Q_1\\ 0\end{bmatrix}\Gamma_{3,t}, \qquad \tilde{\Gamma}_4 = \begin{bmatrix}Q_1\\ 0\end{bmatrix}\Gamma_4, \qquad \tilde{\gamma}_t = \begin{bmatrix}0\\ \gamma_t\end{bmatrix}.$$

Knowing that $\tilde{x}_t = Z'w_t$, we can write Eq. (9) as

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{B}u_t + \tilde{c}_t + \tilde{\epsilon}_t, \tag{10}$$

with

$$\tilde{A} = Z\tilde{\Lambda}^{-1}\tilde{\Omega}Z', \qquad \tilde{B} = Z\tilde{\Lambda}^{-1}\tilde{\Gamma}_2, \qquad \tilde{c}_t = [Z\tilde{\Lambda}^{-1}\tilde{\Gamma}_{3,t} + Z\tilde{\Lambda}^{-1}\tilde{\gamma}_t],$$

and

$$\tilde{\Lambda}^{-1} = \begin{bmatrix}\Lambda_{11}^{-1} & -\Lambda_{11}^{-1}\Lambda_{12}\\ 0 & I\end{bmatrix}, \qquad \tilde{\epsilon}_t = Z\tilde{\Lambda}^{-1}\tilde{\Gamma}_4\in_t.$$

Note that the $\tilde{A}$, $\tilde{B}$, $\tilde{c}_t$, $\tilde{\epsilon}_t$ may depend on θ. In nonpathological cases [Stewart (Ref. 20)], the matrix Λ_{11} will be nonsingular. With Eq. (10), we have replaced the RE in the control model.

4. Computing the Stochastic Optimal Solution

Now that we have the model in the form of Eq. (10), we can derive the optimal solution of the model in Eqs. (1a)–(2b). The optimal solution can be obtained through solving the so-called Riccati equation and tracking equation backward in time,[5]

$$K_t = \tilde{W} + \beta E\{\tilde{A}'K_{t+1}\tilde{A}\} - [\tilde{F} + \beta E\{\tilde{A}'K_{t+1}\tilde{B}\}]$$
$$\times [R + \beta E\{\tilde{B}'K_{t+1}\tilde{B}\}]^{-1}[\beta E\{\tilde{B}'K_{t+1}\tilde{A}\} + \tilde{F}], \tag{11}$$

$$p_t = \beta E\{\tilde{A}'K_{t+1}\tilde{c}_t\} + \beta E\{\tilde{A}'\}p_{t+1} - [\tilde{F} + \beta E\{\tilde{A}'K_{t+1}\tilde{B}\}]$$
$$\times [R + \beta E\{\tilde{B}'K_{t+1}\tilde{B}\}]^{-1}[\beta E\{\tilde{B}'K_{t+1}\tilde{c}_t + \beta E\{\tilde{B}'\}p_{t+1}], \tag{12}$$

with the boundary conditions

$$K_T = \tilde{W}, \qquad p_T = -\tilde{W}\bar{x}_T.$$

Here, $\tilde{W}$ and $\tilde{F}$ are the penalty matrices from the objective function adjusted to conformable size. Once we have backward-integrated these equations and since x_0 is known, we can compute the set of optimal instruments by forward integrating

$$u_t = G_t\tilde{x}_t + g_t, \tag{13}$$

with systems Eq. (10) where

$$G_t = -[R' + \beta E\{\tilde{B}'K_{t+1}\tilde{B}\}]^{-1}[\tilde{F}' + \beta E\{\tilde{B}'K_{t+1}\tilde{A}\}], \tag{14}$$

$$g_t = -[R' + \beta E\{\tilde{B}'K_{t+1}\tilde{B}\}]^{-1}[\beta E\{\tilde{B}'K_{t+1}\tilde{c}_t + \beta E\{\tilde{B}'\}p_{t+1}]. \tag{15}$$

The above equations allow us to solve the set of admissible instruments. The components like $E\{\tilde{A}'K_{t+1}\tilde{B}\}$ capture the effect of the parameter uncertainty on the value instruments. The elements of these components [Magnus and Neudecker (Ref. 21)] are

$$d_{i,j} = \hat{\tilde{a}}_i'K_{t+1}\hat{\tilde{b}}_j + \text{tr}(K_{t+1}\Sigma^{\tilde{b}_j\tilde{a}_i}), \tag{16}$$

where $\hat{\tilde{a}}_i$ is the expected ith column of the matrix $\tilde{A}$ and $\hat{\tilde{b}}_j$ is the expected jth column of the matrix $\tilde{B}$; tr( ) is the trace operator. Hence, through the covariance matrix $\Sigma^{\tilde{b}_j\tilde{a}_i}$, the uncertainty of the parameters on the instruments is captured.

5. Learning Algorithm

As mentioned earlier, the components $\tilde{A}$, $\tilde{B}$, $\tilde{c}_t$ may depend on the unknown parameter vector θ, and we have inserted an estimate $\hat{\theta}_{t|t}$ of this parameter vector in order to be able to solve the RE. The vector $\tilde{\epsilon}_t$ depends also on θ, so we have to assume that

$$E\tilde{\epsilon}_t = 0.$$

[5]See Kendrick (Ref. 7, Chapter 6).

However, the estimate of $\hat{\theta}_{t|t}$ will change over time as new information becomes available or as a consequence of policy reactions in the economy. So, as soon as we have implemented the control u_t, we will get a new realization of the state vector x_{t+1}, which enables us to reestimate the parameter vector obtaining $\hat{\theta}_{t+1|t+1}$.

In the literature, a number of procedures for such learning processes are described (for instance, ordinary least squares learning, filtering, or stochastic approximations). Here, we will apply a Kalman filter to update the estimate $\hat{\theta}_{t|t}$ and the covariance matrix $\hat{\Sigma}^{\theta\theta}_{t|t}$. First, it is necessary to project the covariance matrices to period $t+1$ using observation through period t, which produce the priors

$$\hat{\Sigma}^{xx}_{t+1|t} = f^x_{\theta t}\hat{\Sigma}^{\theta\theta}_{t|t}(f^x_{\theta t})' + \Sigma^{\epsilon\epsilon}, \tag{17}$$

$$\hat{\Sigma}^{\theta x}_{t+1|t} = \hat{\Sigma}^{\theta\theta}_{t|t}(f^x_{\theta t}), \tag{18}$$

$$\hat{\Sigma}^{\theta\theta}_{t+1|t} = \hat{\Sigma}^{\theta\theta}_{t|t}, \tag{19}$$

where

$$f^x_{\theta t} = \sum_{i=1}^{n} e_i x_t' a^i_\theta + \sum_{i=1}^{n} e_i u_t' b^i_\theta + \sum_{i=1}^{n} e_i c^i_\theta. \tag{20}$$

Here, the matrix $f^x_{\theta t}$ is the derivative of the system equations[6] with respect to the vector θ. In addition, we also need an estimate of the state vector, which is

$$\hat{x}_{t+1|t} = A(\hat{\theta}_{t|t})x_t + B(\hat{\theta}_{t|t})u_t + c_t(\hat{\theta}_{t|t}) + \sum_{j=1}^{k} D_j(\hat{\theta}_{t|t})\hat{x}_{t+j|t}. \tag{21}$$

Next, we update the parameter estimate and the covariance matrix for period $t+1$ using observation through period $t+1$, which produces the posterior

$$\hat{\theta}_{t+1|t+1} = \hat{\theta}_{t+1|t} + \hat{\Sigma}^{\theta x}_{t+1|t}(\hat{\Sigma}^{xx}_{t+1|t})^{-1}(x_{t+1} - \hat{x}_{t+1|t}), \tag{22}$$

$$\hat{\Sigma}^{\theta\theta}_{t+1|t+1} = \hat{\Sigma}^{\theta\theta}_{t+1|t} - \hat{\Sigma}^{\theta x}_{t+1|t})(\hat{\Sigma}^{xx}_{t+1|t})^{-1}\hat{\Sigma}^{x\theta}_{t+1|t}, \tag{23}$$

so $\hat{\theta}_{t+1|t+1}$ is the new estimate of the parameter vector and $\hat{\Sigma}^{\theta\theta}_{t+1|t+1}$ is the estimated covariance matrix. Starting with an initial estimates $\hat{\theta}_{0|0}$ and $\hat{\Sigma}^{\theta\theta}_{0|0}$, we can update the parameter vector each time new information on the state of the economy becomes available.

Step 0. Set $t=0$ and compute the estimate $\hat{\theta}_{t|t}$ and its corresponding covariance matrix $\hat{\Sigma}^{\theta\theta}_{t|t}$.

[6]For more detailed information, please refer to Appendices L and M in Kendrick (Ref. 7).

Step 1. Set the iteration counter $\nu = 0$.
Step 2. Set the instruments u_i^ν, $i = \{t, t+1, \dots, T+s-1\}$.
Step 3. Compute γ_i^ν, $i = \{t, t+1, \dots, T+s-1\}$, and compute $\tilde{A}$, $\tilde{B}$, and $\tilde{c}_i$, $\forall i$.
Step 4. Apply a standard LQ optimization method to compute a new set of optimal instruments $u_t^{\nu+1}$ using the equation below in place of Eq. (2a),

$$\tilde{x}_{t+1}^{\nu+1} = \tilde{A}(\hat{\theta}_{t|t})\tilde{x}_t^{\nu+1} + \tilde{B}(\hat{\theta}_{t|t})u_t^{\nu+1} + \tilde{c}(\hat{\theta}_{t|t})_t^\nu.$$

Step 5. Set $\nu = \nu + 1$ and go to Step 2 until convergence is reached on the RE part.
Step 6. Estimate $\hat{\theta}_{t+1|t+1}$ and $\hat{\Sigma}_{t+1|t+1}^{\theta\theta}$ using Eqs. (22)–(23).
Step 7. Set $t = t+1$, and go to Step 1 if $t \Leftarrow T$.

Steps 1 to 5 outline the method for solving the stochastic policy framework for the RE part. Step 6 contains the learning part.

6. Example

In this section, we will present an example of the algorithm described in the previous section. Consider a simple macro model with output x_t, consumption c_t, investment i_t, government expenditures g_t, and taxes τ_t. The problem can then be stated as follows: For the model

$$x_{t+1} = c_{t+1} + i_{t+1} + g_{t+1}, \tag{24}$$

$$c_{t+1} = 0.8(x_t - \tau_t) + 200, \tag{25}$$

$$i_{t+1} = 0.2E_t x_{t+2} - 0.1g_{t+1} + 100 + \epsilon_t, \tag{26}$$

$$g_{t+1} = u_t, \tag{27}$$

$$\tau_{t+1} = 0.25x_{t+1}, \tag{28}$$

with $x_0 = 1500$, find a set of admissible controls $U = \{u_0, u_1, \dots, u_9\}$ to minimize the welfare loss function

$$J_T = (1/2)(x_{12} - 1600)^2 + (1/2)\sum_{t=0}^{11}\{(x_t - 1600)^2 + g_t^2\}. \tag{29}$$

If we reduce the above model to one equation for the output, we get

$$x_{t+1} = 0.6x_t + 0.9u_t + 0.2E_t x_{t+2} + 300 + \epsilon_t, \tag{30}$$

which leads to the augmented system

$$\begin{bmatrix} 1 & -0.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 0.9 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 300 \\ 0 \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}. \tag{31}$$

We will set

$$\Sigma^{\epsilon\epsilon} = 1.$$

Let us assume that the parameter $\theta = 0.9$ is unknown to the policy maker. However, he has the wrong initial estimate

$$\hat{\theta}_{0|0} = [0.8]. \tag{32}$$

Furthermore, let us take as an estimate of the variance

$$\hat{\Sigma}^{\theta\theta}_{0|0} = [0.5], \tag{33}$$

which is also arbitrarily chosen. By applying the QZ factorization, we can compute the QZ decomposition

$$\Lambda = \begin{bmatrix} 1.0822 & -0.9136 \\ 0 & 0.1848 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} 0.7546 & 0.3979 \\ 0 & 0.7952 \end{bmatrix}, \tag{34}$$

$$Z = \begin{bmatrix} 0.8203 & -0.5719 \\ 0.5719 & 0.8203 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.6523 & 0.7580 \\ -0.7580 & 0.6523 \end{bmatrix}; \tag{35}$$

so, the eigenvalues are $\{0.7376/1.0822, 07952/0.1848\}$ and the ordering of the system is such that the unstable root 4.3030 is in the lower-right corner. Due to the fact that θ appears only in the B-matrix, we get the following:

$$\tilde{A} = \begin{bmatrix} 0.2966 & 0.5745 \\ 0.2068 & 0.4006 \end{bmatrix}, \qquad \tilde{B}(\hat{\theta}_{0|0}) = \begin{bmatrix} 0.3955 \\ 0.2758 \end{bmatrix}, \tag{36}$$

and for the initial period,

$$\tilde{c}_0 = \begin{bmatrix} 195.66 \\ 615.09 \end{bmatrix}, \tag{37}$$

$$\tilde{W} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = [1], \qquad \tilde{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{38}$$

$$\tilde{x}_0 = \begin{bmatrix} 1500 \\ 1500 \end{bmatrix}, \qquad \tilde{\bar{x}}_0 = \begin{bmatrix} 1600 \\ 0 \end{bmatrix}. \tag{39}$$

Note that we set $E_0 x_1$ equal to 1500 for the first iteration. In order to deal with the boundary conditions of the RE part of the model, we need the steady state of the system. Unfortunately, the policy maker cannot compute the steady state as the steady state depends on the unknown parameter vector θ. However, based on his initial estimate $\hat{\theta}_{0|0}$, he can make an estimate of the steady state. For $\hat{\theta}_{0|0}$, the steady state of the control vector is

$$u_\infty(\hat{\theta}_{0|0}) = 20.40.$$

Hence, $U^0 = \{20.40, \ldots, 20.40\}$ is a good starting point for the instruments.[7]

For computing the stochastic optimal instruments using Eqs. (11)–(15), we need the expectations[8] of

$$E\{\tilde{B}' K_{t+1} \tilde{B}\} = \hat{\tilde{B}}' K_{t+1} \hat{\tilde{B}} + \mathrm{tr}(K_{t+1} \Sigma_{0|0}^{\tilde{B}\tilde{B}}), \tag{40}$$

$\tilde{B}$ being a vector in this example. Given the fact that

$$\hat{\tilde{B}} = \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} \Lambda_{11}^{-1} \hat{B}, \tag{41}$$

the estimated covariance of $\tilde{B}$ at the initial period will be

$$\hat{\Sigma}_{0|0}^{\tilde{B}\tilde{B}} = \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} \Lambda_{11}^{-1} \hat{\Sigma}_{0|0}^{BB} (\Lambda'_{11})^{-1} \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix}', \tag{42}$$

which is

$$\hat{\Sigma}_{0|0}^{\tilde{B}\tilde{B}} = \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix} \Lambda_{11}^{-1} \hat{\Sigma}_{0|0}^{\theta\theta} (\Lambda'_{11})^{-1} \begin{bmatrix} Z_{11} \\ Z_{12} \end{bmatrix}' = \begin{bmatrix} 0.1222 & 0.0852 \\ 0.0852 & 0.0594 \end{bmatrix}. \tag{43}$$

Now that we have this covariance matrix, we can compute the solution shown in Table 1. The results for the estimate $\hat{\theta}_{t|t}$ and θ are presented in

Table 1. Solution of the LQ optimization model with RE.

t	0	1	2	3	4	5	6
x_t	1500	1548	1571	1580	1584	1586	1585
u_t	37.80	28.89	23.12	21.04	19.95	19.45	19.76
u_∞	20.40	18.86	19.03	19.18	19.20	19.19	19.32
t	7	8	9	10	11	12	
x_t	1587	1587	1587	1583	1580	1573	
u_t	19.08	18.87	18.47	16.95	11.57		
u_∞	19.23	19.20	19.22	19.30	19.30		

[7]See Amman and Kendrick (Ref. 22) for the derivations of the steady state solution.

[8]Due to the fact that θ appears only in the matrix $\tilde{B}$, the other expectation components will be equal to their deterministic value.

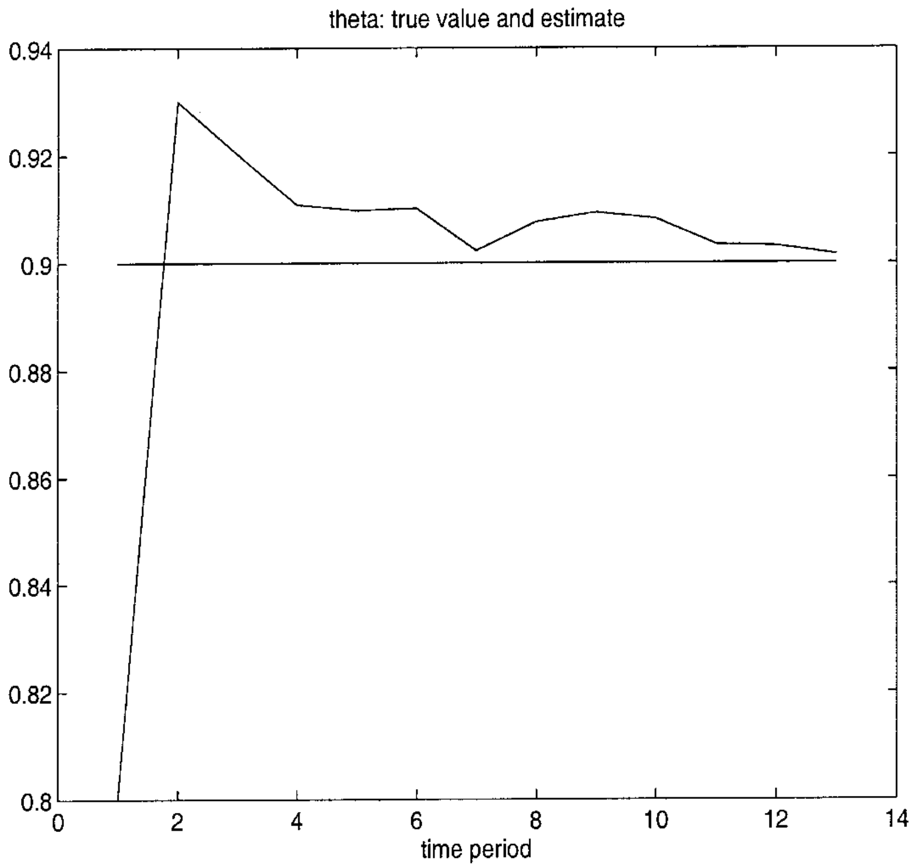

Fig. 1. True value and estimate of theta.

Fig. 1. It is striking how quickly the algorithm is capable of adjusting to the regime switch.

7. Summary

In this paper, we have presented a single-agent stochastic optimization model that allows for rational expectations. Based on the Sim paper, we have used a generalized eigenvalue method for solving the variables that involve unstable roots. By using an iterative scheme, the reduced model can be fitted into a standard linear-quadratic framework that allows us to derive the stochastic optimal policy instruments for the model with rational expectations and learning.

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