ARTICLE IN PRESS

ELSEVIER

Journal of Economic Dynamics & Control ▮▮▮ (▮▮▮▮) ▮▮▮–▮▮▮

JOURNAL OF Economic Dynamics & Control

www.elsevier.com/locate/econbase

Mitigation of the Lucas critique with stochastic control methods

Hans M. Amman[a,*], David A. Kendrick[b]

[a] Executive Board, Technical University of Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

[b] Department of Economics, University of Texas, Austin, TX 78712, USA

Abstract

Lucas (In: Brunner, K., Meltzer, A.H. (Eds.), The Phillips Curve and the Labor Markets, Supplementary Series to the Journal of Monetary Economics, 1976, pp. 19–46) pointed out, that when optimization is performed on a deterministic macro model, the resulting policy may not reflect the true optimal solution. Private agents may react to announced policies and consequently model parameters will start to drift. The aim of this paper is to develop a methodology for deriving an optimal policy in the presence of rational expectations and parameter drift. This drift is captured by a stochastic optimization framework with time-varying parameters. The resulting optimal policy is capable of tracking changes in the parameters due to policy changes. A numerical example illustrates how the methodology provides a way to mitigate the effects of the Lucas critique.

JEL classification: C63; E61

Keywords: Macroeconomics; Rational expectations; Stochastic optimization; Numerical experiments

1. Introduction

In the mid-1970s the use of optimal control techniques for deriving a optimal macroeconomic policy, e.g. Pindyck (1973) and Chow (1975), came under scrutiny. The critique by Lucas (1976) argues that it is difficult to determine optimal macroeconomic policies because the announcement of these policies results in changes in behavior by economic agents and thus changes in the parameters on which the optimal policy was based.

* Corresponding author. Executive Board, Technical University Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

E-mail addresses: h.m.amman@tm.tue.nl (H.M. Amman), kendrick@eco.utexas.edu (D.A. Kendrick).

PII: S0165-1889(02)00115-X

One response to the Lucas critique has been to formulate macroeconomic policy as a game between policy makers and economic agents.[1] This is a useful approach and can provide substantial insights. However, there is another and simpler approach which is employed in this paper. This second approach does not try to predict the game-theory response of economic agents. Rather it attempts to observe and estimate the parameters of the agents behavior in response to policy changes. Thus the decision-maker is always one step (but only one step) behind the economic agents. In a dynamic setting the decision-maker announces a policy, the agent responds and the decision-maker observes the change in behavior and updates the parameters which model that behavior. Those updated parameters are then used in the next period to determine new policy levels. If agent behavior changes with some inertia in response to policy announcements, then the tracking behavior outlined above will yield policies which do well—even relative to the unattainable policies which knows perfectly how the agents will respond.

The methodology we develop here combines (1) the QZ decomposition approach of Sims (1996) to treat the rational expectations, (2) updating methods similar to those recently employed by Sargent (1993, 1999) to update the parameter estimates each period and (3) stochastic control procedures, viz. Amman (1996). This methodology is then applied to a small macroeconomic model in order to compare three scenarios which represent the various policy determination methods discussed above. The model also includes time varying parameters which follow a stochastic process over time. In addition, as described above, the parameters change in response to changes in macroeconomic policy.

We are using here a small IS-LM model in the tradition of Mankiw (1992) and Hall and Taylor (1997). Alternatively, we might have used a dynamic stochastic computable general equilibrium model and converted it to a stochastic control form following the procedures outlined in Smith (1993). This would have put us closer to the tradition of the Lucas critique, viz. Lucas and Sargent (1979); however, it would have involved the use of a more complex model and modeling technique without adding to the central point we want to make. We realize that the previous literature has focused on the ways to develop general equilibrium models that go back to the foundations of economic behavior sufficiently far that changes in policy do not result in changes in the parameters of the model.[2] However, our interest is in the use of simpler and more aggregated models in which changes in policies do result in changes in the parameters and in suggestions for how to determine economic policies by tracking the changes in behavior via estimation updating procedures. Moreover, our focus is not so much on knowing exactly how changes in policies affect changes in the behavioral parameters as it is on using updating estimation procedures to track as quickly and closely as possible the changes in parameters and then using those estimates to determine future policy settings.

[1] Another useful approach which we do not discuss in this paper is the use of policy rules, viz. Feldstein and Stock (1994), Taylor (1993) or Wieland et al. (1999).

[2] Recent literature has studied indeterminacy in general equilibrium models, viz. Benhabib and Farmer (1999); however, we have not encountered indeterminacy in the simple models we have used, so analysis of such cases in beyond the scope of the present inquiry.

While our interest is in the process by which the policy makers can learn about changes in the behavior of the agents, there have been a number of recent papers which model the reverse procedure in which the agents learn about changes in the behavior of the policy makers, viz. Andolfatto and Gomme (1999). These procedures are beyond the scope of the current paper; however, there is no reason why the learning methodologies used in this paper cannot be extended to also include learning by the agents about the behaviors of the policy makers.

The first of the three scenarios is called "Known Parameters". This case is somewhat unrealistic but serves well to illustrate the ideal. The policy makers are assumed to know the original values of the behavioral parameters and to know how changes in policy levels will affect these parameters. However, they are not assumed to know perfectly the small stochastic changes from one period to the next in the parameters which are due to causes other than policy changes. Thus the parameters are not known perfectly except in the original period and parameter estimates are updated each period to keep track of the small stochastic changes as well as the changes due to policy shifts. This produces some uncertainty which is reflected in the variance–covariance matrix of the parameter estimates and this matrix is used in the process of choosing policy levels.

The second scenario is called "Deterministic". In this case policy makers do not know the true values of the parameters in the initial period nor do they know the effects of policy changes on parameter shifts. However, they act as though they know these values perfectly and do not update estimate of parameters means and covariances each period nor do they make use of the parameter covariance matrix in setting policy levels.

The third scenario is called "Learning". This case is similar to the deterministic case above in that policy makers do not know the true values of parameters in the initial time period nor do they know the effects of policy changes on parameter shifts. However, in this case the decision-makers take account of the fact of the parameter uncertainty. They use Kalman filter methods to update and track changes in parameters over time and in the process to provide updated mean and covariance estimates in each time period. Moreover, they make use of the covariance matrix of parameter estimates in deciding on policy levels for the next time period.

We compare the performance of these three approaches in a stochastic environment which includes not only uncertainty about initial parameter estimates (in two of the cases), but also additive noise terms and stochastic time-varying parameters. In doing so we find that the Known Parameters policies perform substantially better than the Deterministic policies. Thus, in this widely used macroeconomic model, if decision-makers knew the initial values of the behavioral parameters and knew how changes in policy would affect these parameters, it would be possible to do a good job of policy determination. In fact, this would be substantially better than policies which ignore the uncertainty in the parameter estimates and ignore the effect of policy changes on shifts in the parameters.

On the other hand we also find that the performance of Known Parameters policies and of Learning policies are quiet similar. Thus even if policy makers do not know exactly the original values of the behavioral parameters and if they do not know how changes in policies will change these parameters, it is possible to develop very good policies. This is accomplished by tracking closely the changes in the parameters from one time period to the next by using a Kalman filter to update parameter estimate means and covariances. Also, the covariance is used in selecting policy levels for the next time period.

Thus, we present an argument in this paper that while game theory is a logical way to address the Lucas critique, an alternative (and simpler) approach may prove to do almost as well. This approach does not attempt to predict how changes in policy will affect agent behavior, but rather only tries to track those changes in behavior and modify policies based on that tracking.

Clearly, the results in this paper are model specific. However, our purpose here is not to claim that the procedure outlined above will entirely eliminate the effects of the Lucas critique. Rather we are providing a methodology which combines elements of the recent thinking of Sims and Sargent within the stochastic control framework in a way that holds substantial promise as a policy determination methodology, which will mitigate the effects of the Lucas critique.

2. Problem statement

2.1. Outline of the methodology

Before describing our model and the mathematics we employ, it is useful to provide a verbal description of the methodology which is used in the Learning scenario. From this basis it is easy to explain, as we proceed, the differences in this methodology from that used in the Known Parameters and Deterministic scenarios.

We begin with a basic IS-LM model in five equations. Some of the equations are substituted out to produce a model with two state variables, two control variables and with forward variables. Then the QZ method from Sims (1996) is used to convert the model into a quadratic-linear, two-state and two-control model which has only backward variables. However, the constant term in this model is a function of the policies chosen and thus of the forward variables. Therefore, it is necessary to solve this model repeatedly, modifying the constant term after each new set of policy variable paths are obtained until convergence is obtained. The policy selection procedure used at each iteration makes use, not only of information about parameter means and states, but also of the variance–covariance matrix of the parameter estimates.

Once the iterations have converged the resulting policy is used and the economy moves one period forward and yields state variables for the next period. These observations are then used in a Kalman filter to update the behavioral parameters of the model. The above process is repeated until all the time periods have been covered.

2.2. The model

Consider a closed economy which can be described by the following four equations

$$Y_t = C(Y_{t-1} - Tax) + I(r_t) + G_{t-1}, \tag{1}$$

$$M^{\mathrm{R}}_{t-1} = L(i_t, Y_t), \tag{2}$$

$$\pi_t = \gamma\pi^{\mathrm{e}}_t + \psi(Y_t - \bar{Y}) + \upsilon_t, \tag{3}$$

$$i_t = r_t + \pi_t + \mu_t, \tag{4}$$

where Y is output, $\bar{Y}$ potential output, C consumption, I investment, G government expenditure, Tax taxation, M^{R} real money supply, L money demand, π inflation, π^{e} expected inflation, i nominal interest rate and r the real interest rate. Both υ and μ are white noise terms.

The above equations describe a dynamic version of the IS-LM model, augmented with a Phillips curve, as can be found in textbooks like Mankiw (1992) and Hall and Taylor (1997). Expectations and random elements enter the model through this Phillips curve and the Fisher effect in Eq. (4). The dynamics enter the model through the consumption function and the *lag structure* of fiscal and monetary policy. We have assumed that both G and M^{R} have lagged effects in the goods market and the money market and that decisions made in $t-1$ have their effects in the next period.

Policy makers are interested in minimizing the discounted welfare loss connected to inflation and output loss as measured by the output gap $(Y_t - \bar{Y})/\bar{Y}$. Hence the loss function becomes

$$\mathrm{E}\{J_T\} = \mathrm{E}\left\{\sum_{t=1}^{T} \beta^t \left\{\phi\left(\frac{Y_t - \bar{Y}}{\bar{Y}}\right)^2 + (1-\phi)\pi_t^2\right\}\right\}, \tag{5}$$

where $\phi \in [0,1]$ reflects the preference of the policy maker for the reduction in output loss versus inflation loss, β is a discount factor and E is the expectations operator. In order to simplify our analysis we will assume that the model in Eqs. (1)–(2) are linear of the form

$$Y_t = bY_{t-1} - b\,\overline{Tax} + \bar{I} - dr_t + G_{t-1}, \tag{6}$$

$$M^{\mathrm{R}}_{t-1} = kY_t - hi_t, \tag{7}$$

where b, k and h are parameters. For convenience we assume that $\bar{I}$ and $\overline{Tax}$ are exogenously determined and time invariant. After some manipulation Eqs. (4), (6) and (7) reduce to the *aggregate demand* curve

$$Y_t = \left(\frac{h}{h+dk}\right)\left(bY_{t-1} - b\,\overline{Tax} + \bar{I} + G_{t-1} + \frac{d}{h}M^{\mathrm{R}}_{t-1} + d\pi_t + d\mu_t\right). \tag{8}$$

Combined with the Phillips curve in (3) this leads to the system

$$\begin{bmatrix}\pi_t\\ Y_t\end{bmatrix} = \begin{bmatrix}0 & \psi\\ \alpha d & 0\end{bmatrix}\begin{bmatrix}\pi_t\\ Y_t\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \alpha b\end{bmatrix}\begin{bmatrix}\pi_{t-1}\\ Y_{t-1}\end{bmatrix} + \begin{bmatrix}\gamma & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\pi^{\mathrm{e}}_t\\ Y^{\mathrm{e}}_t\end{bmatrix}$$
$$+ \begin{bmatrix}0 & 0\\ \alpha & \frac{\alpha d}{h}\end{bmatrix}\begin{bmatrix}G_{t-1}\\ M^{\mathrm{R}}_{t-1}\end{bmatrix} + \begin{bmatrix}-\psi\bar{Y}\\ -\alpha b\,\overline{Tax} + \alpha\bar{I}\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & \alpha d\end{bmatrix}\begin{bmatrix}\upsilon_t\\ \mu_t\end{bmatrix} \tag{9}$$

with $\alpha = h/(h + dk)$. If we define

$$x_t = \begin{bmatrix} \pi_t \\ Y_t \end{bmatrix}, \quad u_t = \begin{bmatrix} G_t \\ M_t^{\mathrm{R}} \end{bmatrix}, \quad g = \begin{bmatrix} -\psi \bar{Y} \\ -\alpha b \overline{Tax} + \alpha \bar{I} \end{bmatrix} \quad \text{and} \quad \xi_t = \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} \tag{10}$$

we can write (9) more compactly

$$x_t = H_0 x_t + H_1 x_{t-1} + H_2 x_t^{\mathrm{e}} + H_3 u_{t-1} + g + H_4 \xi_t, \tag{11}$$

where

$$H_0 = \begin{bmatrix} 0 & \psi \\ \alpha d & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha b \end{bmatrix}, \quad H_2 = \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & 0 \\ \alpha & \frac{\alpha d}{h} \end{bmatrix}, \tag{12}$$

$$g = \begin{bmatrix} -\psi \bar{Y} \\ -\alpha b \overline{Tax} + \alpha \bar{I} \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha d \end{bmatrix}. \tag{13}$$

If $(I - H_0)$ is nonsingular, $A = (I - H_0)^{-1} H_1$, $B = (I - H_0)^{-1} H_3$, $D = (I - H_0)^{-1} H_2$, $V = (I - H_0)^{-1} H_4$, $c = (I - H_0)^{-1} g$ and we shift (11) one period forward we get

$$x_{t+1} = A x_t + B u_t + c + D x_{t+1}^{\mathrm{e}} + V \xi_{t+1}. \tag{14}$$

With classical policy design, we need to assume that the parameters of the above model are known in order to derive the optimal policy. As Lucas (1976) pointed out, this is generally a difficult task as the uncertain parameters of the model may be influenced by the computed optimal policy. The private agents will anticipate the actions of the policy maker and the parameters may start to drift.

To incorporate this element, we allow for parameter drift in the model and include estimates via parameter updating. Therefore, we define the vector of unknown (true) parameters at time t as θ_t. In that case the true model will be

$$x_{t+1} = A(\theta_t) x_t + B(\theta_t) u_t + c(\theta_t) + D(\theta_t) x_{t+1}^{\mathrm{e}} + V(\theta_t) \xi_{t+1}, \tag{15}$$

while the policy maker determines policy based on the equation

$$x_{t+1} = A(\hat{\theta}_t) x_t + B(\hat{\theta}_t) u_t + c(\hat{\theta}_t) + D(\hat{\theta}_t) x_{t+1}^{\mathrm{e}}, \tag{16}$$

where $\hat{\theta}_t$ is the estimate of the unknown parameters in the model. In the next section we will develop a method that derives the optimal solution of Eq. (5) subject to Eq. (15). This method will take into account that the parameters may change over time and that the optimal policy has to reflect the stochastic nature of the model. In terms of the methodology described above we have at this point the two-state–two-control model which includes forward looking variables.

2.3. Solution method

To put the problem formulated in the pervious section on a more general footing, we can define our single-agent stochastic optimization problem as follows (Kendrick, 1981):

Find the set of admissible instruments $U = \{u_0, u_1, \ldots, u_{T-1}\}$ that minimizes the welfare loss function

$$\mathrm{E}\{J_T\} = \mathrm{E}\left\{\beta^T L_T(x_T) + \sum_{t=0}^{T-1} \beta^t L_t(x_t, u_t)\right\} \tag{17}$$

with

$$L_T = \tfrac{1}{2}(x_T - \bar{x}_T)' W (x_T - \bar{x}_T),$$

$$L_t = \tfrac{1}{2}(x_t - \bar{x}_t)' W (x_t - \bar{x}_t) + \tfrac{1}{2}(u_t - \bar{u}_t)' R (u_t - \bar{u}_t) + (x_t - \bar{x}_t)' F (u_t - \bar{u}_t) \tag{18}$$

subject to the model

$$x_{t+1} = A(\theta_t)x_t + B(\theta_t)u_t + c(\theta_t) + \sum_{j=1}^{k} D_j(\theta_t)\mathrm{E}_t x_{t+j} + \varepsilon_t, \tag{19}$$

where the matrix $D_j(\theta_t)$ is a parameter matrix, $\mathrm{E}_t x_{t+j}$ is the expected state for time $t+j$ as formed at time t, k is the maximum lead in the expectations formation[3] and $\varepsilon_t = V(\theta_t)\xi_{t+1}$.

The vector $x_t \in \mathfrak{R}^n$ is the state of the economy at time t and the vector $u_t \in \mathfrak{R}^m$ contains the policy instruments. The initial state of the economy $x_0 \in \mathfrak{R}^n$ is known. The vector $\bar{x}_t \in \mathfrak{R}^n$ and $\bar{u}_t \in \mathfrak{R}^m$ are target values. $W \in \mathfrak{R}^{(n\times n)}$, $R \in \mathfrak{R}^{(m\times m)}$ and $F \in \mathfrak{R}^{(n\times m)}$ are penalty matrices and β is a discount factor; $\varepsilon_t \in \mathfrak{R}^n$ is a white noise vector with $\varepsilon_t \sim$ i.i.d. $N(0, \Sigma^{\varepsilon\varepsilon})$. We assume that $\Sigma^{\varepsilon\varepsilon} \in \mathfrak{R}^{(n\times n)}$, or a reasonable estimate, is known to the policy maker.

A case which is of fundamental theoretical importance, is the situation where the unknown parameter vector is influenced by the choice of the policy instruments. Thus, discretionary policy may be rendered ineffective due to the fact that the policy instruments influence the parameter vector θ_t, or in general terms

$$\theta_t = G(u_t, u_{t+1}, \ldots, u_{T-1}). \tag{20}$$

Therefore, it is interesting to investigate the extent to which our stochastic optimization approach is able to learn parameter drift quickly enough to make discretionary policy effective. Learning is introduced into the stochastic optimization framework by the unknown parameter vector $\theta_t \in \mathfrak{R}^p$ which is determined through a learning strategy. We assume that the parameter vector follows a random walk

$$\theta_{t+1} = \theta_t + \eta_t, \tag{21}$$

where $\eta_t \sim$ i.i.d. $N(0, \Sigma^{\eta\eta})$. We assume that $\Sigma^{\eta\eta} \in \mathfrak{R}^{(p\times p)}$, or a reasonable estimate, is known to the policy maker. This means that the policy maker will be unaware of the functional relationship of Eq. (20), but knows that at least some of the parameters in the model may change over time due to unforseen causes.

[3] See also Amman (1996).

The above model *without* rational expectations can be solved using the procedures outlined in Kendrick (1981) and Tucci (1997). In order to compute the admissible set of instruments, we have to transform the rational expectations model to a model that only contains backward looking variables. In two earlier paper, Amman and Kendrick (1999a, b), we described how the control model with rational expectations can be solved using Sims (1996) approach.[4] After solving for the rational expectations part of the model we obtain

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{B}u_t + \tilde{c}_t + \tilde{\varepsilon}_t, \tag{22}$$

where $\tilde{c}_t$ depends on the value of future controls. Through this term the future controls have impact on both the current state vector and the expectations of future events. Due to this forward looking component we need to apply an iterative procedure that gives us the solution of Eq. (22). This is established by iterating on

$$\tilde{x}_{t+1}^{v+1} = \tilde{A}\tilde{x}_t^{v+1} + \tilde{B}u_t^{v+1} + \tilde{c}_t^v + \tilde{\varepsilon}_t, \tag{23}$$

where v is the iteration counter.[5] When we have the model in the form of Eq. (23) we can derive the optimal solution of the model in Eqs. (17)–(19).

Next, in the steps of our methodology we describe the procedure used to compute the optimal control at each iteration while making use of the parameter estimates and covariance matrix. The optimal solution at period t can be obtained by solving the Riccati equation[6] and tracking equation backward in time (using the assumptions that $F = 0$ and $\bar{u} = 0$)

$$\begin{aligned} K_j = {} & \tilde{W} + \beta \mathrm{E}\{\tilde{A}' K_{j+1}\tilde{A}\} - [\tilde{F} + \beta \mathrm{E}\{\tilde{A}' K_{j+1}\tilde{B}\}] \\ & \times [R + \beta \mathrm{E}\{\tilde{B}' K_{j+1}\tilde{B}\}]^{-1}[\beta \mathrm{E}\{\tilde{B}' K_{j+1}\tilde{A}\} + \tilde{F}], \end{aligned} \tag{24}$$

$$\begin{aligned} p_j^v = {} & \beta \mathrm{E}\{\tilde{A}' K_{j+1}\tilde{c}_t^v\} + \beta \mathrm{E}\{\tilde{A}'\} p_{j+1}^v - [\tilde{F} + \beta \mathrm{E}\{\tilde{A}' K_{j+1}\tilde{B}\}] \\ & \times [R + \beta \mathrm{E}\{\tilde{B}' K_{j+1}\tilde{B}\}]^{-1}[\beta \mathrm{E}\{\tilde{B}' K_{t+1}\tilde{c}_t^v\} + \beta \mathrm{E}\{\tilde{B}'\} p_{j+1}^v] \end{aligned} \tag{25}$$

with $j = \{T-1, \ldots, t\}$ and the boundary conditions

$$K_T = \tilde{W},$$

$$p_T^v = -\tilde{W}\tilde{x}_T.$$

Note that the Riccati matrices K_j do not change from iteration to iteration, but that the Riccati vectors p_j^v do change. The penalty matrices $\tilde{W}$ and $\tilde{F}$ are the penalty matrices from the objective function adjusted to conformable size. Once we have integrated these equations backward in time and given that x_0^{v+1} is known, we can compute the

[4] Sims' method is basically an extention of the well-known method of Blanchard and Kahn (1980).

[5] For this iterative procedure we also need the steady state of the model. The procedure to compute this steady state can be found in Amman and Kendrick (1998) and Amman and Neudecker (1997).

[6] See Kendrick (1981), Chapter 6 and Amman and Kendrick (1999c) Appendix A. Please note that the vector λ_j in these equations is set to zero.

set of optimal instruments by forward integration of

$$u_j^{v+1} = G_j \tilde{x}_j^{v+1} + g_j^v \tag{26}$$

and the systems equations (22), where now $j = \{t, \ldots, T-1\}$. The matrices G_j and g_j^v are equal to

$$G_j = -[R' + \beta \mathrm{E}\{\tilde{B}' K_{j+1} \tilde{B}\}]^{-1} [\tilde{F}' + \beta \mathrm{E}\{\tilde{B}' K_{j+1} \tilde{A}\}], \tag{27}$$

$$g_j^v = -[R' + \beta \mathrm{E}\{\tilde{B}' K_{j+1} \tilde{B}\}]^{-1} [\beta \mathrm{E}\{\tilde{B}' K_{j+1} \tilde{c}_j^v\} + \beta \mathrm{E}\{\tilde{B}'\} p_{j+1}^v]. \tag{28}$$

The above equations allow us to solve for the policy variables. The components like $\mathrm{E}\{\tilde{A}' K_{j+1} \tilde{B}\}$ capture the effect of the parameter uncertainty on the value of instruments. For instance, the matrix $\mathrm{E}\{\tilde{B}' K_{j+1} \tilde{A}\}$ can be computed by[7]

$$\begin{aligned} \mathrm{E}\{vec(\tilde{B}' K_{j+1} \tilde{A})\} &= (vec(K_{j+1})' \otimes I_{nm})(I_n \otimes K_{nn} \otimes I_m) \\ &\quad \times (K_{nn} \otimes K_{mn})\, vec(vec(\hat{\tilde{B}})\, vec(\hat{\tilde{A}})' + \hat{\Sigma}_{t|t}^{\tilde{B}\tilde{A}}), \end{aligned} \tag{29}$$

where $\mathrm{E}\{\tilde{A}\} = \hat{\tilde{A}}$, $\mathrm{E}\{\tilde{B}\} = \hat{\tilde{B}}$ and $vec()$ is the vec operator (cf. Magnus and Neudecker, 1999). The effect of uncertainty in the parameters on the instruments is captured through the covariance matrix $\hat{\Sigma}_{t|t}^{\tilde{B}\tilde{A}}$.[8] Hence, the derived optimal policy is stochastic and takes into account the fact that certain parameters in the model matrices may change over time. As soon as we have the expected terms of Eqs. (24)–(28), we can solve for the set of admissible instruments through backward induction. We have to repeat this procedure until we have convergence on the controls in Eq. (23).

3. Learning the unknown parameters

Next we turn our attention to the use of the Kalman filter to update the parameter estimates and covariance matrix after the new information on the state variables, the state of the economy, is observed. As in Eq. (22) the terms $\tilde{A}$, $\tilde{B}$, $\tilde{c}_t$ may depend on the unknown parameter vector θ_t and we have inserted an estimate $\hat{\theta}_t$ of this parameter vector in order to be able to solve the rational expectations.[9] The vector $\tilde{\varepsilon}_t$ also depends on θ_t, so we have assumed that $\mathrm{E}\tilde{\varepsilon}_t = 0$.

However, the estimate of θ_t, $\hat{\theta}_t$, will change over time as new information becomes available or as a consequence of policy reactions. So, as soon as we have implemented the control u_t, we will get a new realization of the state vector x_{t+1}, which enables us to re-estimate the parameter vector obtaining $\hat{\theta}_{t+1}$. In the literature a number of procedures for such learning processes are described. For instance, ordinary least squares learning,

[7] Eq. (29) is derived in Appendix A.

[8] For simulation studies of the effect of parameter uncertainty in macroeconomic models see Fair (1984).

[9] In this paper we do not discuss how to obtain an initial estimate of the parameters of the rational expectations model. Some references on this issue are Swamy and Tinsley (1980), Wallace (1980) and Wickens (1982).

filtering, or stochastic approximations (Ljung et al., 1992; Sargent, 1993, 1999). Here, we will apply a Kalman filter to update the estimate $\hat{\theta}_t$ and the covariance matrix $\hat{\Sigma}^{\theta\theta}_{t|t}$.

First, it is necessary to project the covariance matrices to period $t+1$ using observation through period t, which produce the priors [10]

$$\hat{\Sigma}^{xx}_{t+1|t} = f^x_{\theta t}\hat{\Sigma}^{\theta\theta}_{t|t}(f^x_{\theta t})' + \Sigma^{\varepsilon\varepsilon}, \tag{30}$$

$$\hat{\Sigma}^{\theta x}_{t+1|t} = \hat{\Sigma}^{\theta\theta}_{t|t}(f^x_{\theta t}), \tag{31}$$

$$\hat{\Sigma}^{\theta\theta}_{t+1|t} = \hat{\Sigma}^{\theta\theta}_{t|t} + \Sigma^{\eta\eta}, \tag{32}$$

where

$$f^x_{\theta t} = \sum_{i=1}^{n} e_i x_t' a^i_\theta + \sum_{i=1}^{n} e_i u_t' b^i_\theta + \sum_{i=1}^{n} e_i c^i_\theta. \tag{33}$$

Here the matrix $f^x_{\theta t}$ is the derivative of the system equations [11] with respect to the vector θ_t and e_i is a vector of zeros except for a one in the ith position. In addition we also need the expected value of the state vector, which is

$$\hat{x}_{t+1|t} = A(\hat{\theta}_t)x_t + B(\hat{\theta}_t)u_t + c(\hat{\theta}_t)_t + \sum_{j=1}^{k} D_j(\hat{\theta}_t)\hat{x}_{t+j|t}. \tag{34}$$

Next we update the parameter estimate and the covariance matrix for period $t+1$ using observation through period $t+1$, which produces the posterior

$$\hat{\theta}_{t+1} = \hat{\theta}_t + \hat{\Sigma}^{\theta x}_{t+1|t}(\hat{\Sigma}^{xx}_{t+1|t})^{-1}(x_{t+1} - \hat{x}_{t+1|t}), \tag{35}$$

$$\hat{\Sigma}^{\theta\theta}_{t+1|t+1} = \hat{\Sigma}^{\theta\theta}_{t+1|t} - \hat{\Sigma}^{\theta x}_{t+1|t}(\hat{\Sigma}^{xx}_{t+1|t})^{-1}\hat{\Sigma}^{x\theta}_{t+1|t}, \tag{36}$$

so $\hat{\theta}_{t+1}$ is the new estimate of the parameter vector and $\hat{\Sigma}^{\theta\theta}_{t+1|t+1}$ the estimated covariance matrix. Starting with an initial estimates $\hat{\theta}_0$ and $\hat{\Sigma}^{\theta\theta}_{0|0}$ we can update the parameter vector each time new information on the state of the economy becomes available.

In summary, our proposed methodology is as follows:

Algorithm for computing optimal instruments

Step 0: Set $t=0$ and compute an estimate of $\hat{\theta}_t$ and its corresponding covariance matrix $\hat{\Sigma}^{\theta\theta}_{t|t}$.

Step 1: Set the iteration counter $v=0$.

Step 2: Set the instruments u^v_j, $j=\{t, t+1, \ldots, T-1\}$.

Step 3: Compute $\tilde{A}$, $\tilde{B}$ and $\forall j \tilde{c}_j$ for $j=\{t, t+1, \ldots, T-1\}$.

[10] See Kendrick (1981, Chapter 10, p. 102).

[11] For more detailed information please refer to Appendices L and M in Kendrick (1981).

Step 4: Apply stochastic optimization method in Eqs. (24)–(28) to compute a new set of optimal instruments $\forall j u_j^{v+1}$.

Step 5: Set the iteration counter $v = v + 1$ and go to Step 2 until convergence is reached on the instruments.

Step 6: Integrate the systems equation (22) one time step forward thereby obtaining the state variables for the next period.

Step 7: Estimate the parameter values $\hat{\theta}_{t+1}$ and covariance matrix $\hat{\Sigma}^{\theta\theta}_{t+1|t+1}$ using Eqs. (35)–(36).

Step 8: Set $t = t + 1$ and go to Step 1 if $t <= T$.

Hence, Steps 1–5 outline the method for solving the stochastic policy framework for the rational expectations part. Steps 6 and 7 contains the learning part.[12]

4. A numerical example

Lets return to the example of Section 2 and assume that we have the following set of parameter and exogenous variables: $\psi = 0.002$, $b = 0.9$, $d = 25$, $\gamma = 0.9$, $h = 100$, $k = 1$, $\beta = 1$, $\bar{Y} = 500$, $\overline{Tax} = 50$, $\bar{I} = 100$, $\phi = \frac{1}{2}$ and $T = 12$. The matrix F in Eq. (18) is set to zero and the matrix R from that same equation is set to a diagonal matrix with small numbers on the diagonals, $R = 10^{-6} I$. The problem can then be stated as:

Find the set of admissible instruments $U = \{u_0, u_1, \ldots, u_{12}\}$ that minimizes the welfare loss function

$$\mathrm{E}\{J_{12}\} = E\left\{\sum_{t=0}^{12}\left\{\frac{1}{2}\left(\frac{Y_t - 500}{500}\right)^2 + \frac{1}{2}\pi_t^2\right\}\right\} \tag{37}$$

subject to the model

$$\begin{bmatrix} \pi_{t+1} \\ Y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0.0015 \\ 0 & 0.7500 \end{bmatrix} \begin{bmatrix} \pi_t \\ Y_t \end{bmatrix} + \begin{bmatrix} 0.0017 & 0.0004 \\ 0.8333 & 0.2083 \end{bmatrix} \begin{bmatrix} G_t \\ M_t^{\mathrm{R}} \end{bmatrix} + \begin{bmatrix} -0.9500 \\ 25.0000 \end{bmatrix}$$
$$+ \begin{bmatrix} 0.9375 & 0 \\ 18.7500 & 0 \end{bmatrix} \begin{bmatrix} \pi_{t+1}^{\mathrm{e}} \\ Y_{t+1}^{\mathrm{e}} \end{bmatrix} + \begin{bmatrix} 1.0417 & 0.0417 \\ 20.8333 & 20.8333 \end{bmatrix} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} \tag{38}$$

with the initial state

$$x_0 = \begin{bmatrix} \pi_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 500 \end{bmatrix}, \tag{39}$$

which is the numeric version of Eq. (14) with x_t, u_t and ξ_t as defined in Eq. (10). We will set

$$\Sigma^{\xi\xi} = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix} \tag{40}$$

[12] A Matlab implementation is available on request by email from the corresponding author amman@fee.uva.nl.

consequently,

$$\Sigma^{\varepsilon\varepsilon} = V\Sigma^{\xi\xi}V' = \begin{bmatrix} 1.08681\ 10^{-4} & 2.25694\ 10^{-3} \\ 2.25694\ 10^{-3} & 8.68056\ 10^{-2} \end{bmatrix}. \tag{41}$$

Now lets assume that the impact of the real money supply is uncertain and that the two coefficient in the second column of B are time varying and unknown to the policy maker. This means that the coefficients

$$\theta_0 = \begin{bmatrix} 0.0004 \\ 0.2083 \end{bmatrix} \tag{42}$$

will change over time, possibly due to the Lucas critique, and have to be (re)estimated by the policy maker. Furthermore, assume that the policy maker has a wrong initial estimate of this parameter vector which is

$$\hat{\theta}_0 = \begin{bmatrix} 0.0004 \\ 0.3000 \end{bmatrix}. \tag{43}$$

The vector $\hat{\theta}_0$ is chosen in such a fashion that the effect of the real money supply on output is initially overestimated. Furthermore, lets take as an estimate of the variance

$$\hat{\Sigma}^{\theta\theta}_{0|0} = \begin{bmatrix} 10^{-8} & 0 \\ 0 & 0.0625 \end{bmatrix}. \tag{44}$$

Due to Eq. (21) the true parameter θ_t will follow a random walk in subsequent periods. In order to add this feature to the example we will set

$$\Sigma^{\eta\eta} = \begin{bmatrix} 1.610^{-13} & 0 \\ 0 & 0.01 \end{bmatrix}. \tag{45}$$

To mimic the Lucas critique in our example we will assume that the random vector θ_t is influenced by the real money supply in the following way:

$$\theta_{t+1} = \theta_t + \left(\frac{\Delta M_t^{\mathrm{R}}}{M_t^{\mathrm{R}}}\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \theta_t + \eta_t, \tag{46}$$

which means that an increase in the real money has an increasing effect on inflation and a decreasing effect on output. The policy maker will be unaware of this relationship and will try to estimate θ_t based on the random walk assumption.

There is still one loose end which needs to be taken care of. In order to compute the stochastic optimal instruments, using the algorithm from the previous section, we need the expectations of $\mathrm{E}\{\tilde{B}'K_{j+1}\tilde{B})$ which is Eq. (A.10) from the Appendix A.[13]

[13] Due to the fact that θ only appears in the $\tilde{B}$, the other expectation components will be equal to their mean value. How to compute $\tilde{A}$, $\tilde{B}$ and $\tilde{c}$ is described in Amman and Kendrick (1999a).

For this purpose we need to compute $\hat{\Sigma}^{\tilde{B}\tilde{B}}_{t|t}$ from $\hat{\Sigma}^{BB}_{t|t}$. Using $\hat{\Sigma}^{\theta\theta}_{t|t}$ we get for the initial period

$$\hat{\Sigma}^{BB}_{0|0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-8} & 0 \\ 0 & 0 & 0 & 0.0625 \end{bmatrix}. \tag{47}$$

With the help of Eq. (A.10) and (A.12) and knowing that

$$\hat{\tilde{B}}(\hat{\theta}_0) = \begin{bmatrix} -0.0444 & -0.0160 \\ 0 & 0 \end{bmatrix} \tag{48}$$

gives us

$$\hat{\Sigma}^{\tilde{B}\tilde{B}}_{0|0} = 10^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1778 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{49}$$

and

$$\mathrm{E}\{\tilde{B}' K_1 \tilde{B}\} = 10^{-3} \begin{bmatrix} 0.9876 & 0.1777 \\ 0.1777 & 0.1742 \end{bmatrix}, \tag{50}$$

which give us the values required for the optimization as captured in Eqs. (22)–(28).

The results: With the model structure above one can address the question of how best to do macroeconomic policy formulation in an environment in which agents may change their behavior over time and as well as in response to economic policy pronouncements.

Consider first the unrealistic case in which the policy makers know exactly the component of the drift in the parameters which is due to policy changes. As discussed above, we call this scenario Known Parameters. This is accomplished in the model by correcting the estimation of the parameters each time period when the updating occurs, that is

$$\hat{\theta}_{t+1} = \hat{\theta}_t + \hat{\Sigma}^{\theta x}_{t+1|t}(\hat{\Sigma}^{xx}_{t+1|t})^{-1}(x_{t+1} - \hat{x}_{t+1|t}) + \left(\frac{\Delta M^{\mathrm{R}}_t}{M^{\mathrm{R}}_t}\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \theta_t, \tag{51}$$

so the policy maker is aware of the parameter shifts described in Eq. (46). Also, the parameters in this scenario are stochastic and time varying due to η_t. In the face of the fact that parameter drift may be occurring in the economy, policy may be determined in a deterministic manner treating all parameters as known and constant—i.e. the Deterministic scenario. In this case the true values of the parameters drift but the policy maker does not know what these values are and does not modify his or

Table 1
Comparison of the Known Parameters and Deterministic Scenarios

Scenario	$E\{J_T\}$	J_T^{min}	J_T^{max}	σ_{J_T}
Known Parameters	1.638	1.280	1.808	0.089
Deterministic	1.675	1.386	3.591	0.193

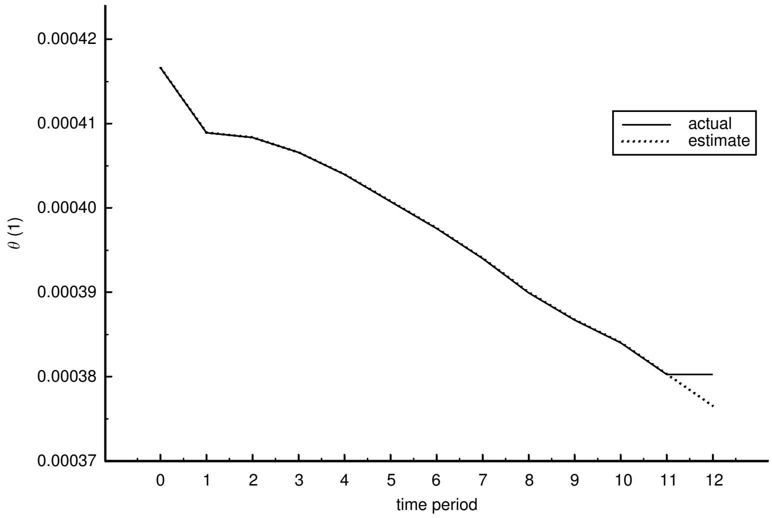

Fig. 1. Known Parameters ($\theta(1)$).

her policy in any way because of the stochastic elements of the model. One can then compare these two cases to asks whether or not knowing the parameter drift makes a difference. In order to do this we performed 1000 Monte Carlo runs with the model above and obtained the results shown in Table 1.

In both of these cases the true parameters are drifting over time and also changing in response to shifts in policy values; however, in the Known Parameters case the policy maker knows the effect of policy changes on the drift and in the Deterministic case he or she uses parameter values which are not correct, but treats those values as though they are (i) correct and (ii) that there is no uncertainty attached to the parameter estimate. Not surprisingly the average criterion values is better at 1.638 for the known parameter case than at 1.675 for the deterministic case. Also from the max and min comparisons one can see that there is considerably more variability in the deterministic case than in the known parameter case and this is confirmed in the last column which shows the standard deviation of the criterion values across the 1000 Monte Carlo runs.

As an example of the nature of these results compare Figs. 1 and 2 which show the values of parameter $\theta(1)$ in the two cases. The actual value of the parameter drifts down in both cases but changes by different amounts because the use of the policy variables is not the same in the two cases. In contrast the estimated values tracks the

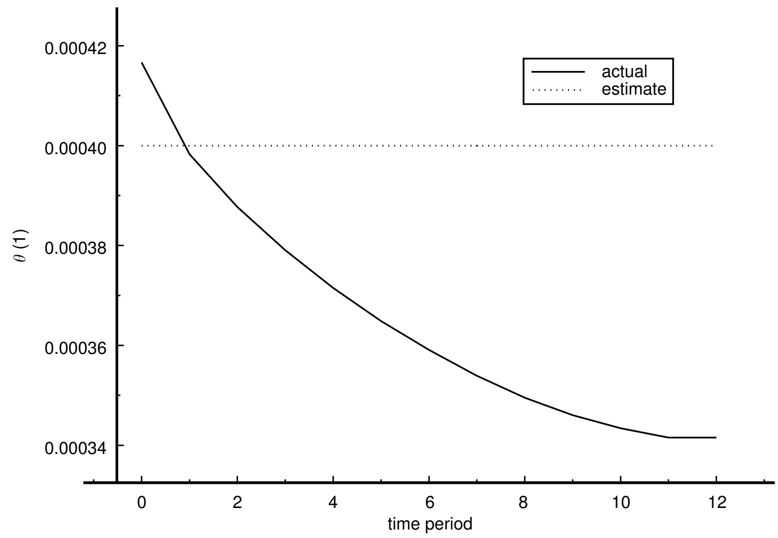

Fig. 2. Deterministic ($\theta(1)$).

Table 2
Comparison of the Deterministic and Learning Scenarios

Scenario	$E\{J_T\}$	$J_T^{\min}$	$J_T^{\max}$	σ_{J_T}
Deterministic	1.675	1.386	3.591	0.193
Learning	1.646	1.292	1.817	0.090

actual very closely in the Known Parameter case but does not track and indeed does not change in the deterministic case.

So for the example at hand the scenario of Known Parameters is better than the scenario of ignoring the fact of the unknown parameters in the Deterministic case. Thus the Lucas critique is valid in this case. Ignoring the fact that the parameters are drifting and changing in response to policy pronouncements results in higher loss functions for the performance of the economy.

Would it be possible to do better with some other policy formulation method? One alternative approach would be to use game theory between the policy makers and the agents. Another, and the one which is examined here, is for the policy makers to (1) treat seriously the fact that they are using parameter estimates rather than the true parameter values when making policy and (2) update parameter estimates as the true values of the parameters drift over time and shift in response to policy announcements. As discussed above, we call this scenario the Learning approach. One can then asks whether the learning approach is indeed better than the deterministic case. This comparison is provided in Table 2.

The average value of the criterion function over the 1000 Monte Carlo runs is better at 1.646 for the Learning case than at 1.675 for the Deterministic case. Also, the

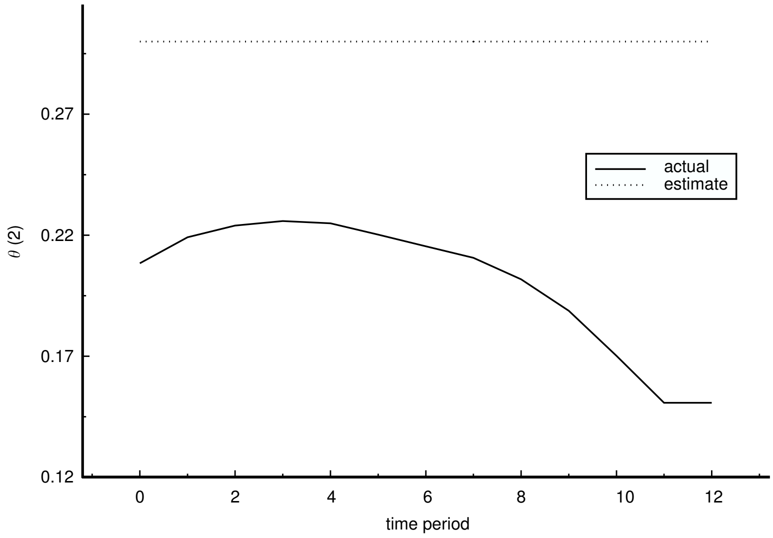

Fig. 3. Deterministic ($\theta(2)$).

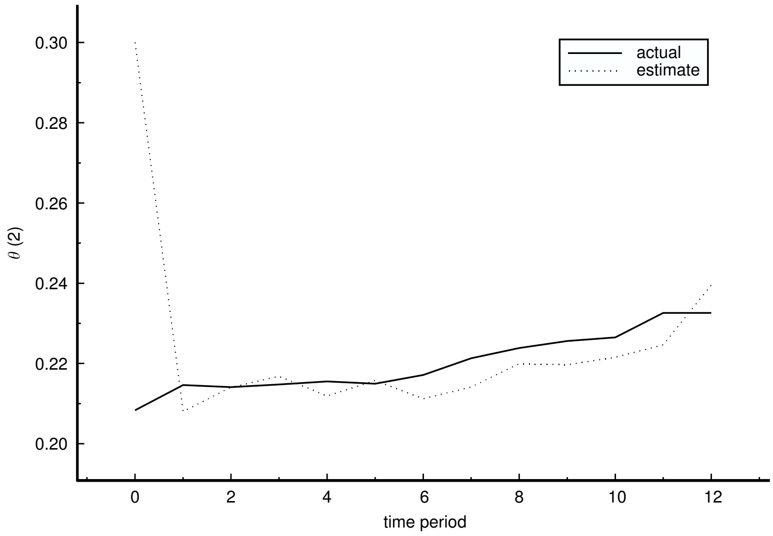

Fig. 4. Learning ($\theta(2)$).

variability is lower for the learning case than the deterministic case as is shown in the remaining three columns of Table 2. Thus, for the case at hand, it is better to take account of the uncertainty in the parameters when determining policy in an environment where parameters are changing. Figs. 3 and 4 show the true and estimated parameter estimates for the deterministic and learning scenarios—this time for parameter $\theta(2)$.

Table 3
Comparison of the Known Parameters and Learning Scenarios

Scenario	$E\{J_T\}$	$J_T^{\min}$	$J_T^{\max}$	σ_{J_T}
1: Learning	1.646	1.292	1.817	0.090
2: Known Parameters	1.638	1.280	1.808	0.089

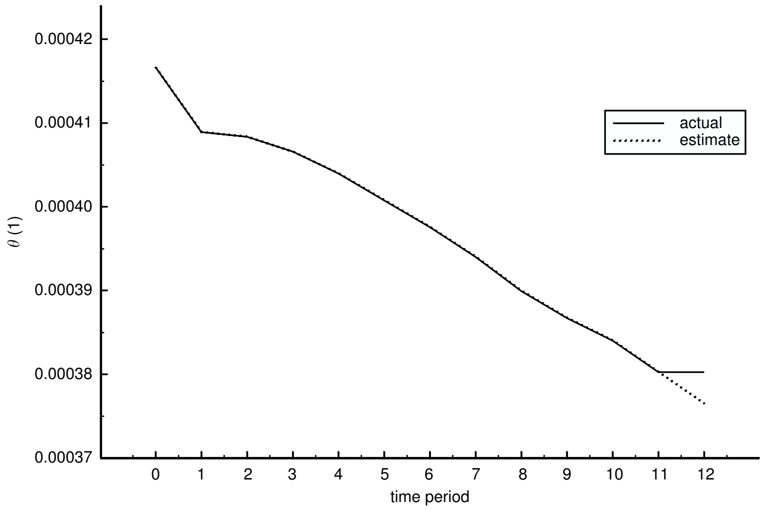

Fig. 5. Known Paramters ($\theta(1)$).

As before, in the deterministic case there is no change in the parameter estimate in Fig. 3. In contrast Fig. 4 shows that there is a very rapid learning of the $\theta(2)$ parameter in the learning scenario and that this parameter estimate then tracks closely the drift in the actual parameter over time.

Finally, one can ask whether the Lucas critique still carries heavy weight when policy makers treat seriously the uncertainty of parameter estimates in determining policy levels. Table 3 provides an indication of how this question may be answered by providing a comparison of the Known Parameters and the Learning scenarios.

There is little difference between these two cases in either the means in the first column or in the indicators of variance in the remaining three columns. Thus for this commonly used small macroeconomic model, the effects of the Lucas critique are substantially mitigated if one uses policy determination methods which treat the uncertainty in parameter estimates seriously and track the drifting parameters over time by using Kalman filter estimators.

A comparison of Figs. 5 and 6 show that the Known Parameters scenario provides a better tracking of $\theta(1)$ than does the Learning scenario. However, a comparison of Figs. 7 and 8 shows that the Learning method does almost as well at tracking $\theta(2)$ as does the Known Parameters method.

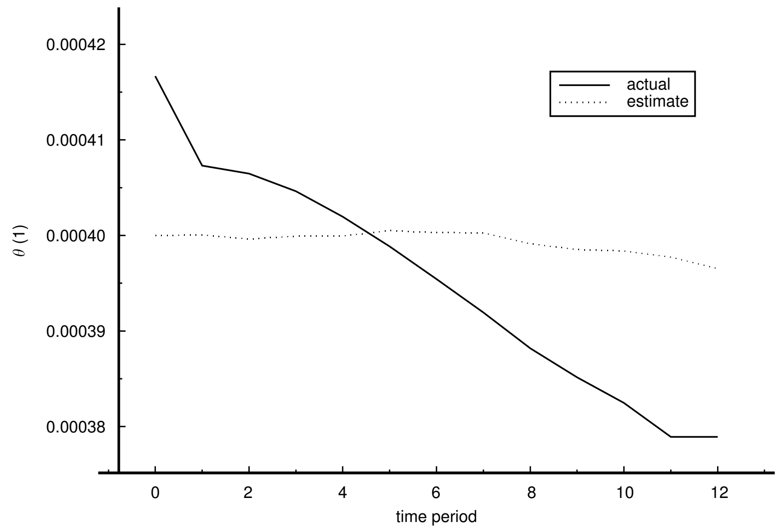

Fig. 6. Learning ($\theta(1)$).

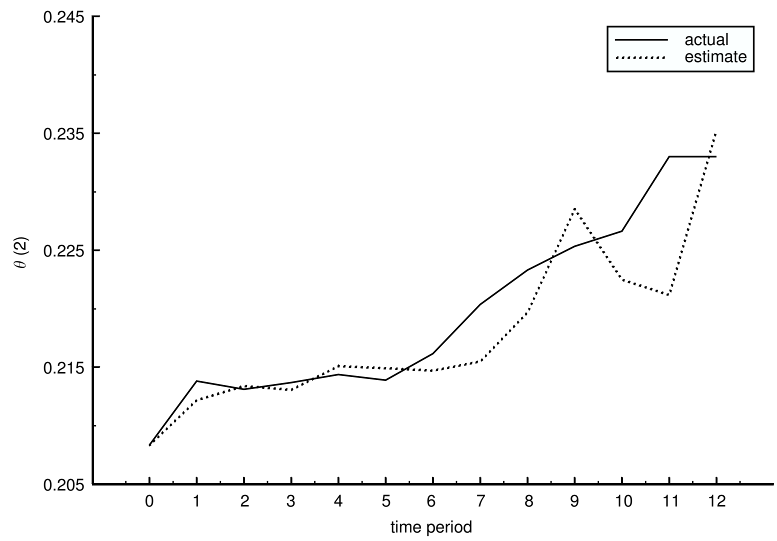

Fig. 7. Known Parameters ($\theta(2)$).

5. Summary

Our purpose in this article is to provide a simpler methodology than game theory as a vehicle to mitigate the effects of the "Lucas critique". To do this we bring together ideas from Sims on solving rational expectations models and from Sargent on updating

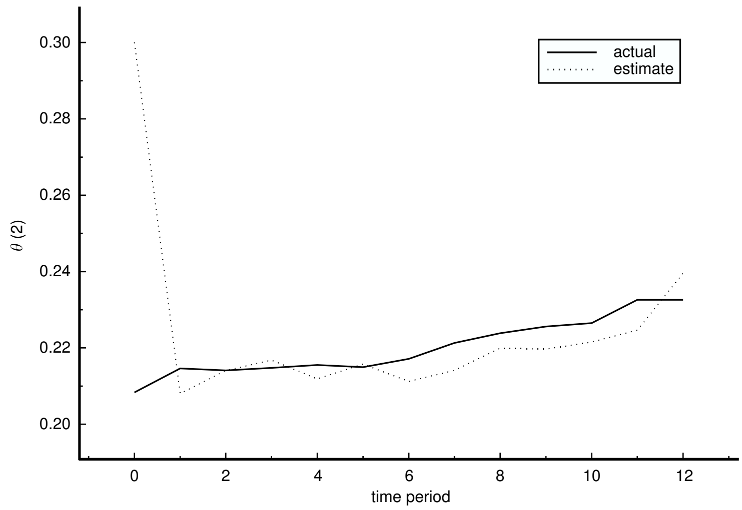

Fig. 8. Learning ($\theta(2)$).

parameters and insert these into a comprehensive stochastic control framework. We then apply this methodology to a widely used form of the IS-LM model to which we add an effect of policy changes on the behavioral parameters of the model.

Using this basis we compare three macroeconomic policy scenarios. In the first of these three scenarios we adopt the unlikely proposition that policy makers know the exact effect of policy changes on behavioral parameters. We then compare the results of this policy to one in which the policy makers do not know these effects but act as though they know the behavioral parameters perfectly. In this setting, the Lucas critique is confirmed and the second policy is substantially worse than the first.

Then we introduce a Learning scenario in which policy makers update the behavioral parameters each period and thus attempt to track the drifts in behavior which are caused by the changes in policy. When we compare this Learning scenario to the first scenario in which policy makers know the effects of policy changes on parameter drift, we find that there is not much difference in performance. Thus this methodology offers the promise that it may substantially mitigate the effects of the Lucas critique.

Acknowledgements

We would like to thank Heinz Neudecker for useful comments on an earlier version of the paper and a referee for a number of helpful comments.

Appendix A

In this appendix we will derive an alternative form for $\mathrm{E}\{\tilde{B}' K_{j+1} \tilde{A}\}$ as required in Eqs. (24)–(28). $\tilde{A}$ and $\tilde{B}$ are random matrices as they may contain time varying

ARTICLE IN PRESS

parameters. K_{j+1} is a known matrix. For convenience we drop the projection subscript j of the matrix K. For three conformable matrices X, Y and Z the following holds:

$$vec(XYZ) = (Z' \otimes X)\, vec(Y) \tag{A.1}$$

vec being the vec operator.[14] Given (A.1) the following holds:

$$\mathrm{E}\{vec(\tilde{B}'K\tilde{A})\} = \mathrm{E}\{(\tilde{A}' \otimes \tilde{B}')vec(K)\} \tag{A.2}$$

reapplying the rule in (A.1)–(A.2) we get

$$\mathrm{E}\{vec(\tilde{B}'K\tilde{A})\} = \mathrm{E}\{(vec(K)' \otimes I_{nm})\, vec(\tilde{A}' \otimes \tilde{B}')\}, \tag{A.3}$$

where I_{nm} is an identity matrix of order $(nm \times nm)$. The last term on the right-hand side can be written as

$$vec(\tilde{A}' \otimes \tilde{B}') = (I_n \otimes K_{nn} \otimes I_m)(vec(\tilde{A}') \otimes vec(\tilde{B}')), \tag{A.4}$$

where K_{mn} is the commutation matrix. Knowing that

$$vec(\tilde{A}') \otimes vec(\tilde{B}') = vec(vec(\tilde{B}')\, vec(\tilde{A}')')$$

we have (Magnus and Neudecker, 1999, p. 30)

$$\begin{aligned} vec(\tilde{A}' \otimes \tilde{B}') &= (I_n \otimes K_{nn} \otimes I_m)(vec(vec(\tilde{B}')\, vec(\tilde{A}')')) \\ &\quad (I_n \otimes K_{nn} \otimes I_m)(vec(K_{mn}\, vec(\tilde{B})(K_{nn}\, vec(\tilde{A}))')) \\ &\quad (I_n \otimes K_{nn} \otimes I_m)(vec(K_{mn}\, vec(\tilde{B})\, vec(\tilde{A})'K_{nn})) \end{aligned} \tag{A.5}$$

because $K_{nn} = K'_{nn}$. Inserting (A.5) into (A.3) results in

$$\begin{aligned} \mathrm{E}\{vec(\tilde{A}'K\tilde{B})\} &= \mathrm{E}\{(vec(K)' \otimes I_{nm})(I_n \otimes K_{nn} \otimes I_m) \\ &\quad \times (vec(K_{mn}vec(\tilde{B})vec(\tilde{A})'K_{nn}))\} \end{aligned} \tag{A.6}$$

carrying through the expectation operator gives us

$$\begin{aligned} \mathrm{E}\{vec(\tilde{A}'K\tilde{B})\} &= (vec(K)' \otimes I_{nm})(I_n \otimes K_{nn} \otimes I_m) \\ &\quad (vec(K_{mn}\mathrm{E}\{vec(\tilde{B})\, vec(\tilde{A})'\}K_{nn})). \end{aligned} \tag{A.7}$$

If we have two random matrices X and Y with $\mathrm{E}(vec(X)) = vec(\hat{X})$ and $\mathrm{E}(vec(Y)) = vec(\hat{Y})$ we know that their covariance is equal to

$$\begin{aligned} \hat{\Sigma}^{XY} &= E(vec(X) - vec(\hat{X}))(vec(Y) - vec(\hat{Y}))' \\ &\quad E(vec(X)\, vec(Y)') - vec(\hat{X})\, vec(\hat{Y})'. \end{aligned} \tag{A.8}$$

[14] See Magnus and Neudecker (1999, p. 30).

Applied to (A.7) leads to

$$
\begin{aligned}
\mathrm{E}\{vec(\tilde{A}'K\tilde{B})\} &= (vec(K)' \otimes I_{nm})(I_n \otimes K_{nn} \otimes I_m) \\
&\quad \times vec(K_{mn}(vec(\hat{\tilde{B}})\,vec(\hat{\tilde{A}})' + \hat{\Sigma}^{\tilde{B}\tilde{A}})K_{nn}) \\
&= (vec(K)' \otimes I_{nm})(I_n \otimes K_{nn} \otimes I_m) \\
&\quad \times vec((K_{nn} \otimes K_{mn})\,vec(vec(\hat{\tilde{B}})\,vec(\hat{\tilde{A}})' + \hat{\Sigma}^{\tilde{B}\tilde{A}})) \\
&= (vec(K)' \otimes I_{nm})(I_n \otimes K_{nn} \otimes I_m) \\
&\quad \times (K_{nn} \otimes K_{mn})\,vec(vec(\hat{\tilde{B}})\,vec(\hat{\tilde{A}})' + \hat{\Sigma}^{\tilde{B}\tilde{A}}),
\end{aligned} \tag{A.9}
$$

where $\mathrm{E}\{vec(\tilde{A})\} = vec(\hat{\tilde{A}})$ and $\mathrm{E}\{vec(\tilde{B})\} = vec(\hat{\tilde{B}})$. In the example in Section 5 we need to evaluate the term $\mathrm{E}\{\tilde{B}'K_{j+1}\tilde{B}\}$ at time t for $j = \{T, \ldots, t\}$. Based on the above equation we get

$$
\begin{aligned}
\mathrm{E}\{vec(\tilde{B}'K_{j+1}\tilde{B})\} &= (vec(K_{j+1})' \otimes I_{mm})(I_n \otimes K_{nm} \otimes I_m) \\
&\quad \times (K_{nm} \otimes K_{mn})\,vec(vec(\hat{\tilde{B}})\,vec(\hat{\tilde{B}})' + \hat{\Sigma}^{\tilde{B}\tilde{B}}_{t|t}).
\end{aligned} \tag{A.10}
$$

Hence, we need $vec(\hat{\Sigma}^{\tilde{B}\tilde{B}}_{t|t})$ to be able to compute (A.11). The matrix $\hat{\tilde{B}}$ is computed from

$$
\hat{\tilde{B}} = Z\tilde{\Lambda}^{-1}Q_1\hat{B} \tag{A.11}
$$

of which we know its numerical value from Eq. (48). From (A.11) it is easy to see that

$$
\begin{aligned}
& vec(\hat{\tilde{B}}) = (I_n \otimes Z\tilde{\Lambda}^{-1}Q_1)\,vec(\hat{B}) \\
& \quad \Rightarrow \hat{\Sigma}^{\tilde{B}\tilde{B}}_{t|t} = (I_n \otimes Z\tilde{\Lambda}^{-1}Q_1)\hat{\Sigma}^{BB}_{t|t}(I_n \otimes Z\tilde{\Lambda}^{-1}Q_1)' \\
& \quad \Rightarrow vec(\hat{\Sigma}^{\tilde{B}\tilde{B}}_{t|t}) = (I_n \otimes Z\tilde{\Lambda}^{-1}Q_1) \otimes (I_n \otimes Z\tilde{\Lambda}^{-1}Q_1)\,vec(\hat{\Sigma}^{BB}_{t|t})
\end{aligned} \tag{A.12}
$$

the matrices Z, $\tilde{\Lambda}$ and Q_1 originate from the QZ decomposition to remove the rational expectations variables from the model, see Amman and Kendrick (1999b). Knowing that in this example

$$
\hat{\Sigma}^{BB}_{t|t} = \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \hat{\Sigma}^{\theta\theta}_{t|t} \end{bmatrix} \tag{A.13}
$$

we can compute the expected terms of (A.10). The matrix $\hat{\Sigma}_{t|t}^{BB}$ is semi positive definite due to the fact that the first two elements of $\hat{B}$ are deterministic.

References

Amman, H.M., 1996. Numerical methods for dynamic optimization problems. In: Amman, H.M., Kendrick, D.A., Rust, J. (Eds.), Handbook of Computational Economics. North-Holland Publishers, Amsterdam, pp. 579–618.

Amman, H.M., Kendrick, D.A., 1998. Computing the steady state of the linear quadratic optimization model with rational expectations. Economics Letters 58, 185–191.

Amman, H.M., Kendrick, D.A., 1999a. Stochastic policy design in a learning environment with rational expectations. Journal of Optimization Theory and Applications 106, 509–520.

Amman, H.M., Kendrick, D.A., 1999b. Linear-quadratic optimization for models with rational expectations. Macroeconomic Dynamics 3, 534–543.

Amman, H.M., Kendrick, D.A., 1999c. Matrix methods for solving nonlinear dynamic optimization models. In: Heijmans, R.J., Pollock, D.S.G., Satorra, A. (Eds.), Innovations in Multivariate Statistical Analysis. Kluwer Academic Publishers, Dordrecht, pp. 257–276.

Amman, H.M., Neudecker, H., 1997. Numerical solution methods of the algebraic matrix Riccati equation. Journal of Economic Dynamics and Control 21, 363–370.

Andolfatto, D., Gomme, P., 1999. Monetary policy regimes and beliefs. Working Paper 9905R, Federal Reserve Bank of Cleveland, Cleveland, Ohio.

Benhabib, J., Farmer, R.E.A., 1999. Indeterminacy and sunspots in macroeconomics. Handbook of Macroeconomics, Vol. 1a, Ch. 6. Elsevier Science, North-Holland Publishers, Amsterdam.

Blanchard, O.J., Kahn, C.M., 1980. The solution of linear difference models under rational expectations. Econometrica 48, 1305–1311.

Chow, G.C., 1975. Analysis and Control of Dynamic Economic Systems. Wiley, New York.

Fair, R., 1984. Specification, Estimation and Analysis of Macroeconometric Models. Harvard University Press, Cambridge, MA.

Feldstein, M., Stock, J.H., 1994. The use of a monetary aggregate to target nominal GDP. In: Mankiw, N.G. (Ed.), Monetary Policy. University of Chicago Press, Chicago, pp. 7–70.

Hall, R.E., Taylor, J.B., 1997. Macroeconomics, 5th Edition. W.W. Norton and Company, New York.

Kendrick, D.A., 1981. Stochastic Control for Economic Models. McGraw-Hill, New York.

Ljung, L., Pflug, G., Walk, H., 1992. Stochastic Approximation and Optimization of Random Systems. Birkhäuser, Berlin.

Lucas, R.E., 1976. Econometric policy evaluation: a critique. In: Brunner, K., Meltzer, A.H. (Eds.), The Phillips Curve and the Labor Markets, Supplementary Series to the Journal of Monetary Economics, pp. 19–46. North-Holland, Amsterdam.

Lucas, R.E., Sargent, T.J., 1979. After Keynesian macroeconomics. Reserve Bank of Minneapolis Quarterly Review 3, 1–16.

Magnus, J.R., Neudecker, H., 1999. Matrix Differential Calculus with Applications in Statistics and Econometrics, Revised Edition. Wiley, New York.

Mankiw, N.G., 1992. Macroeconomics. Worth Publishers, New York.

Pindyck, R.S., 1973. Optimal Planning for Economic Stabilization. North-Holland, Amsterdam.

Sargent, T.J., 1993. Bounded Rationality in Macroeconomics. Clarendon Press, Oxford.

Sargent, T.J., 1999. The Conquest of American Inflation. Princeton University Press, Princeton, NJ.

Sims, C.A., 1996. Solving linear rational expectations models. Research Paper, Department of Economics, Yale University.

Smith, B.G., 1993. Optimal regional economic policies using computable general equilibrium models. Ph.D. Dissertation, Department of Economics, The University of Texas, Austin, TX.

Swamy, P.A.V.B., Tinsley, P., 1980. Linear prediction and estimation methods for regression models with stationary stochastic coefficients. Journal of Econometrics 12, 103–142.

Taylor, J.B., 1993. Discretion versus policy rules in practice. Carnegie–Rochester Conference Series on Public Policy 39, 195–214.

Tucci, M.P., 1997. Adaptive control in the presence of time varying parameters. Journal of Economic Dynamics and Control 22, 39–47.

Wallace, K.F., 1980. The econometric implications of the rational expectations hypothesis. Econometrica 48, 49–73.

Wickens, M., 1982. The efficient estimation of econometric models with rational expectations. Review of Economic Studies 49, 55–67.

Wieland, V., Levin, A., Williams, J.C., 1999. The performance of forward-looking monetary policy rules under model uncertainty. Paper presented at the Fifth International Conference of the Society of Computational Economics, Boston College, Chestnut Hill, MA, USA, June 24–26.