# Abstract

The aim of this project thesis is to give a brief overview of some applications of string theory to black hole physics, particularly their entropy. This involves connections between the degeneracy of solitonic BPS states in string theory and the macroscopic entropy of black holes in supergravity theories. Research into this area was kickstarted by a seminal paper by Strominger and Vafa on bound D1-D5 states of Type II string theory compactified on $K3 \times S^1$ [1] and has subsequently been expanded upon by Dabholkar, Sen et al. [2, 3, 4]

We will first develop the basic building blocks of string theory required for these constructions, in particular the concepts of low-energy supergravity actions, compactification and duality. In order to understand solitonic BPS states, we will also introduce the BPS bound, D-branes and heterotic string theory. We then give a few examples of possible BPS states and briefly sketch the computation of their microstate degeneracy.

The heterotic string compactified on a torus admits an intuitive half-BPS state with a tower of massive states, forming a black hole at strong coupling. We will then look at some aspects of the D1-D5-P system in $D = 5$ and $D = 4$, the latter of which also contains magnetic charges described by a Kaluza-Klein monopole. We will compare the expressions for the entropy obtained in these cases, all of which are approximated by the Cardy formula in conformal field theory:

$$d(N, c) \sim \exp\left(2\pi\sqrt{\frac{Nc}{6}}\right)$$

# Contents

# 1. Macroscopic Entropy of Black Holes

## 1.1. Classical Black Holes

Black holes arise as solutions of the field equations of general relativity, the Einstein field equations (with $c = 1$):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{1.1}$$

These equations of motion are derived from the Einstein-Hilbert-Maxwell action:

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g}\, R + \int d^D x \sqrt{-g}\, \mathcal{L}_{Matter} \tag{1.2}$$

The Schwarzschild black hole in D dimensions is a static solution of the vacuum equations

$$R_{\mu\nu} = 0 \tag{1.3}$$

where the spacetime manifold is taken to have the topology $\mathcal{M}_2 \times S^{D-2}$. The general Ansatz for the metric is then [5]

$$ds^2 = -f^2(r) dt^2 + g^2(r) dr^2 + r^2 d\Omega^2_{D-2} \tag{1.4}$$

The equations of motion imply that

$$f(r) = g^{-1}(r) = \left(1 - \frac{C}{r^{D-3}}\right) \tag{1.5}$$

where we require that $D \geq 4$. Note that in lower dimensions, there also exist black hole solutions, namely the BTZ black hole in (2+1) spacetime dimensions [6], as well as a black hole analogue in (1+1) dimensions [7]. These models are of interest to study the properties of black holes and possible approaches to quantum gravity.

The integration constant $C$ in (1.5) is related to the mass $M$ by

$$M = C \, \frac{(D-2) A_{D-2}}{16\pi} \tag{1.6}$$

with $G_N = 1$, and where $A_{D-2}$ is the area of the unit $(D-2)$-sphere:

$$A_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)} \tag{1.7}$$

To find the general solution of the Reissner-Nordström black hole in dimension D, we take the matter content to be determined by the Maxwell-Lagrangian density

$$S_{MW} = -\frac{1}{2} \int F \wedge \star F \quad \rightarrow \quad \mathcal{L}_{MW} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{1.8}$$

We now introduce an electromagnetic U(1) gauge potential given by the 1-form:

$$A = h(r)\,dt \tag{1.9}$$

The field strength is defined by $F = dA$, giving a radial electric field:

$$F = -\frac{\partial h}{\partial r}\,dt \wedge dr \tag{1.10}$$

The electric charge of the field is then

$$Q = \frac{1}{2A_{D-2}} \int_{S^{D-2}} F^{\mu\nu} dS_{\mu\nu} \tag{1.11}$$

where the electromagnetic field around a point charge is given by

$$F^{0k} = \frac{Q}{4\pi} \frac{x^k}{r^{(D-1)}}\,. \tag{1.12}$$

The equations of motion then determine $f(r), g(r), h(r)$ and the charge $Q$ as follows:

$$f(r) = g^{-1}(r) = \left(1 - \frac{C}{r^{D-3}} + \frac{\tilde{C}^2}{r^{2(D-3)}}\right)^{1/2} \tag{1.13}$$

$$h(r) = \pm\left(\frac{1}{2}\frac{D-2}{D-3}\right)^{1/2} \frac{\tilde{C}}{r^{D-3}} \tag{1.14}$$

$$Q = \pm\tilde{C}\left(\frac{(D-2)(D-3)}{2}\right)^{1/2} \tag{1.15}$$

For example, the familiar Reissner-Nordström metric in $D = 4$ is then:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{1.16}$$

This geometry has two distinct horizons, an event horizon $r_+$ and a Cauchy horizon:

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{1.17}$$

We exclude the case where the square root is imaginary in order to have well-defined horizons, thus avoiding the problem of a possible naked singularity. Therefore we have:

$$M \geq |Q| \tag{1.18}$$

When the bound is saturated by $M = |Q|$, the metric describes an extremal black hole. Note that this condition is similar in character to the BPS bound that we will define later, which has lead to the conjecture that macroscopic properties of extremal black holes, particularly the entropy, are described by the statistical entropy $S_{stat} \sim \log d(N)$ of a BPS soliton state in compactified string theory models. Note that we can also introduce magnetic charge $P$ if we have a magnetic field $F_{ij} = (P/4\pi)\sin\theta$, yielding the metric:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{1.19}$$

## 1.2. Black hole thermodynamics

### 1.2.1. Bekenstein-Hawking entropy

Recall that the thermodynamic partition function of a system is given by [8]

$$Z = Tr\Big(e^{-\beta H}\Big) \tag{1.20}$$

and that evolution of a quantum system over time $t$ is given by the propagator:

$$U(t) = e^{-iHt} \tag{1.21}$$

The trace in (1.20) then implies that a system with temperature $T = \beta^{-1}$ is periodic in Euclidean time $\tau = it$ with a period $\beta$. We can therefore calculate the entropy of a Schwarzschild black hole by considering analytic continuation of the Schwarzschild metric to Euclidean time via a Wick rotation $t \to i\tau$. We approximate the near-horizon metric of the Schwarzschild black hole around $r = r_H$ or equivalently $\rho = 0$ with $r \equiv r_H(1+\rho^2)$, leaving out terms in $d\Omega^2$ that describe the spherical part of the metric:

$$ds^2_{\tau,\rho} \approx 4r_H^2\Big(\rho^2(\frac{d\tau}{2r_H})^2 + d\rho^2\Big) \tag{1.22}$$

We find that by requiring this to give a regular metric at the horizon $\rho = 0$, i.e. a flat plane in polar coordinates $ds^2 = r^2 d\theta^2 + dr^2$, this must satisfy a periodicity condition:

$$\tau \sim \tau + 2\pi = \tau + \beta \qquad (\beta = d\tau) \quad \to \quad \frac{d\tau}{2r_H} = 2\pi \tag{1.23}$$

Inserting the horizon radius $r_H = 2M$, we see that the black hole has a finite temperature:

$$T = \frac{1}{8\pi M} \tag{1.24}$$

The Bekenstein-Hawking entropy of a black hole in terms of the area of its horizon is now:

$$S_{BH} = \frac{A}{4} \tag{1.25}$$

As an example, the extremal Reissner-Nordström black hole in $D = 4$ has the metric

$$ds^2 = -\Big(1 - \frac{r_0}{r}\Big)^2 dt^2 + \Big(1 - \frac{r_0}{r}\Big)^{-2} dr^2 + r^2 d\Omega_2^2 \tag{1.26}$$

where the radius of the horizon is given by $r_0 = M = |Q|$. The entropy is then given by:

$$A = 4\pi r_0^2 \quad \to \quad S = \pi r_0^2 = \pi Q^2 \tag{1.27}$$

### 1.2.2. Wald entropy

From string theory, we expect "stringy" corrections to the effective action with higher derivative terms, involving functions of the Riemann tensor and other bosonic fields:

$$S = \frac{1}{16\pi}\int d^Dx\,\sqrt{-g}\,R \quad \rightarrow \quad S = \frac{1}{16\pi}\int d^Dx\,\sqrt{-g}\,\Big(R + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + f(R,F) + ...\Big) \tag{1.28}$$

For such actions, Wald derived a generalized expression for the entropy:

$$S_W = 2\pi\int_{\rho^2}\sqrt{h}\,d^2\Omega\,\frac{\delta S}{\delta R_{\mu\nu\alpha\beta}}\epsilon^{\mu\alpha}\epsilon^{\nu\beta} \tag{1.29}$$

where the integral is over the horizon surface $\rho^2$, $\epsilon^{\mu\nu}$ is binormal to the horizon (e.g. for a $S^2$-horizon with two normal directions $t, r$ we have $\epsilon_{tr} = -1$), and $h$ is the induced metric on the horizon. For the Einstein-Hilbert action, we get the Bekenstein-Hawking entropy.

If we have a theory with multiple gauge fields, we naturally get multiple conserved charges. The entropy can in this case be written in the general form $S = 2\pi\sqrt{\Delta}$, where $\Delta$ must be an invariant under the symmetry group of the charges. An example is given by black hole states in $\mathcal{N} = 8$ supergravity in $D = 5$, or equivalently, Type IIB string theory on $T^4 \times S^1$ with $Q_1$ D1-branes, $Q_5$ D5-branes and $n$ units of Kaluza-Klein momentum - described further in later chapters - where we have for the entropy [8]:

$$S = 2\pi\sqrt{\Delta} = 2\pi\sqrt{Q_1Q_5n} \tag{1.30}$$

Another example we briefly mention is given by a construction in Type IIA theory on $T^6$ with three sets of D2-branes wrapping orthogonal two-tori of $T^6$ and D6-branes wrapping the entire torus $T^6$. This system has four electric charges $Q_1...Q_4$, yielding the entropy

$$S = 2\pi\sqrt{\Delta} = 2\pi\sqrt{Q_1Q_2Q_3Q_4} \tag{1.31}$$

We can extend this system by adding $P_1$ D0-branes, which - as explained later - are the magnetic duals of D6-branes. The corresponding entropy is then given by:

$$S = 2\pi\sqrt{\Delta} = 2\pi\sqrt{Q_1Q_2Q_3Q_4 - \frac{1}{4}{P_1}^2{Q_1}^2} \tag{1.32}$$

As an example, we consider a half-BPS black hole with electric charge vector $Q$ and zero magnetic charge $P$ as discussed in [4]. Generally, for quarter-BPS states with non-zero magnetic charge (dyonic states), the Wald entropy is given by the approximation:

$$S_W(Q,P) = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2} + ... \tag{1.33}$$

For $P = 0$, the leading term in this expression would vanish, implying a vanishing horizon area. If we include higher derivative terms in the action, we get a non-vanishing entropy:

$$S_W(Q) = 4\pi\sqrt{\frac{Q^2}{2}} \tag{1.34}$$

# 2. String Theory

We begin with a short review of the basic aspects of string theory [8]:

## 2.1. The bosonic string

Denoting a point particle as a 0-brane, a string as a 1-brane, and further extended objects as p-branes where $p \geq 2$, the motion of a general p-brane is described by the action

$$S_p = -T_p \int d\mu_p \tag{2.1}$$

where $T_p$ is the p-brane tension and $d\mu_p$ is the $(p+1)$-dimensional volume form. Varying this action, we get the equations of motion of a relativistic point particle for $p = 0$, the equations of motion of a relativistic bosonic string for $p = 1$, and so on. In particular, the action for $p = 1$ is the familiar Nambu-Goto action:

$$S_{NG} = -T \int d\tau d\sigma \sqrt{-|G_{\alpha\beta}|}\,, \qquad G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu\,, \qquad X^\mu = (X^0, ..., X^{D-1}) \tag{2.2}$$

Note that the integral in this action is the area of the string world sheet - varying the action thus minimizes the world-sheet area, analogous to the case $p = 0$ where varying $S = -m \int ds$ minimizes the length of the particle worldline.

A classically equivalent and covariant formulation of the bosonic string is given by the string sigma-model action (also known as the Polyakov action [9])

$$S_\sigma = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} g_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu \tag{2.3}$$

where $h_{\alpha\beta}$ is the 2-dimensional world-sheet metric - acting as an auxiliary field - and $g_{\mu\nu}$ is the metric of the D-dimensional target space. The equation of motion for the coordinates $X^\mu$ is a wave equation:

$$\Box X^\mu = \partial_\alpha \partial^\alpha X^\mu = \eta^{\alpha\beta} \partial_\alpha \partial_\beta X^\mu = 0 \tag{2.4}$$

Variation with respect to $h_{\alpha\beta}$ constrains the energy-momentum tensor, such that $T_{\alpha\beta} = 0$. The action has several symmetries:

- Poincaré transformations: $\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu$
- Reparametrization of the world-sheet coordinates: $\sigma^\alpha \to f^\alpha(\sigma)$
- Weyl transformations: $h_{\alpha\beta} \to e^{\phi(\sigma,\tau)} h_{\alpha\beta}$

The latter two symmetries allow us to adopt the conformal gauge $h_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta}$. In light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$ the equations of motion then simplify to $\partial_+\partial_- X^\mu = 0$.

The boundary conditions imposed on $X^\mu$ give open and closed strings:

- Closed strings:
  - Periodic boundary conditions: $X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma)$
- Open strings:
  - Neumann boundary conditions: $\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0$
  - Dirichlet boundary conditions: $X^\mu(\tau, 0) = X^\mu(\tau, \pi) = c^\mu$

We can expand the string coordinates $X^\mu$ into a Fourier series with the oscillator modes $\alpha^\mu{}_n, \tilde{\alpha}^\mu{}_n$ for the right- and left-moving parts. The mode operators have to satisfy the following commutation relations upon quantization:

$$[\alpha^\mu_m, \alpha^{\nu\dagger}_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^{\nu\dagger}_n] = im\eta^{\mu\nu}\delta_{m+n,0} \tag{2.5}$$

From the mode expansion of the energy-momentum tensor we get the Virasoro generators

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} : \alpha_{m-n}\alpha_n :, \qquad \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} : \tilde{\alpha}_{m-n}\tilde{\alpha}_n : \tag{2.6}$$

which satisfy the Virasoro algebra with the central charge $c$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{2.7}$$

The last term is called the central extension or conformal anomaly, related to the fact that string theory can be understood as a two-dimensional conformal field theory (CFT) on the worldsheet. The Virasoro generators act on physical states $|\phi\rangle$ as $L_m |\phi\rangle = 0$ for $m > 0$ and $(L_0 - a)|\phi\rangle = 0$, where $a$ is the normal ordering constant. For the closed string we also have to impose the level-matching condition $(L_0 - \tilde{L}_0)|\phi\rangle = 0$. Requiring the absence of negative-norm states of the quantized string and Lorentz invariance of the spectrum then fixes the dimension of the target space to $D = 26$ and the normal ordering constant to $a = -1$.

The spectrum of the open string contains a tachyonic ground state $|0; k\rangle$, a massless vector boson $\alpha^i_{-1}|0; k\rangle$ and further massive excitations. The ground state of the closed string is again a tachyon $|0; k\rangle$, but the massless state is a tensor state $|\Omega^{ij}\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0; k\rangle$. This state can be decomposed into parts, containing:

- $|\Omega^{(ij)}\rangle$, a symmetric, traceless spin-2 particle (a graviton field $g^{ij}$)
- $\delta_{ij}|\Omega^{ij}\rangle$, a scalar field (a dilaton field $\phi$)
- $|\Omega^{[ij]}\rangle$, an antisymmetric, traceless two-form field (a Kalb-Ramond field $B^{ij}$)

## 2.2. The supersymmetric string

### 2.2.1. Worldsheet supersymmetry

Introducing fermionic fields into the theory, we have the supersymmetric string action:

$$S = -\frac{T}{2} \int d^2\sigma (\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \tag{2.8}$$

This action is invariant under the supersymmetry transformations:

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \,, \quad \delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \tag{2.9}$$

where $\epsilon$ is an infinitesimal Majorana spinor consisting of Grassmann numbers. Associated with the global world-sheet supersymmetry is a conserved supercurrent $J_A^\alpha$. After choosing light-cone gauge, the possible boundary conditions for $\psi^\mu$ are:

$$\text{Ramond (R):} \quad \psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi} \,, \quad \text{Neveu-Schwarz (NS):} \quad \psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi} \tag{2.10}$$

Note that Ramond boundary conditions lead to spacetime fermions, while Neveu-Schwarz boundary conditions give spacetime bosons. The mode expansion of $\psi^\mu$ is given in terms of the oscillator modes $d_n^\mu$ (R) or $b_n^\mu$ (NS). For the closed string, the boundary conditions can be chosen for left- and right-movers independently. This gives rise to the four sectors R-R, R-NS, NS-R, NS-NS - the mixed sectors contain fermionic states, while the pure sectors contain bosonic states. The modes of $\psi^\mu$ are quantized by anticommutation relations:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \,, \quad \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \tag{2.11}$$

The Virasoro generators $L_m$ get an extra contribution from the modes of $\psi^\mu$ and the modes of the supercurrent $F_m$ and $G_r$ are given by sums of $\alpha_n$ with $d_n$ and $b_r$, for Ramond and Neveu-Schwarz sectors respectively. The super-Virasoro algebra follows by extending the Virasoro algebra with (anti-)commutation relations for $L_m$ and $F_m$ or $G_r$. Absence of negative norm states for the superstring implies $D = 10$. The spectrum of the open string in the NS-sector has a tachyonic ground state, while the first excited states is a massless vector boson. The ground state spinor in the R-sector is massless, while the excited states are massive. The GSO projection, consisting of discarding all states with negative G-parity $(-1)^{F_{NS}} = -1$ from the NS sector and choosing a chirality in the R-sector determined by $\Gamma_{11}(-1)^{F_R} = \pm 1$, where $\Gamma_{11}$ is the ten-dimensional equivalent of the chirality operator $\gamma_5$ and $F_{NS,R}$ are the number operators, eliminates the tachyon from the spectrum and gives unbroken spacetime supersymmetry with $\mathcal{N} = 1$ for open strings.

In the light-cone formalism, the physical degrees of freedom are contained in the transverse components of the modes, therefore giving $D - 2 = 8$ degrees of freedom. This is in agreement with the Majorana-Weyl representation of the spinors, where the spinor components are real and the spinor has definite chirality and thus is an eigenstate of $\Gamma_{11}$.

In the case of the closed string, left-movers and right-movers are combined into the four sectors mentioned above. In the left- and right-moving R-sector the ground states can be choosen to have opposite or same chirality, which corresponds to Type IIA and Type IIB string theory, respectively. The combined degrees of freedom give $8 \times 8 = 64$ physical states in each of the massless sectors, which can be interpreted in the different sectors as follows [8]:

- NS-NS sector: dilaton (one state), antisymmetric two-form gauge field (28 states), symmetric traceless tensor (35 states)
- NS-R and R-NS sectors: spin-1/2 dilatino, spin-3/2 gravitino
- R-R sector:
  1. Type IIA (opposite chirality): vector gauge field (eight states), three-form gauge field (56 states)
  2. Type IIB (same chirality): scalar gauge field (one state), two-form gauge field (28 states), four-form gauge field with self-dual ($\star F = F$) field strength (35 states)

### 2.2.2. Spacetime supersymmetry

The previous results are also achieved in the Green-Schwarz formalism, which has the advantage of having manifest spacetime supersymmetry. Rewriting the string theory as maps of the string worldsheet into superspace rather than Lorentzian spacetime, we take as basic fields the worldsheet coordinates $X^\mu(\sigma,\tau)$ and anticommuting spinor coordinates $\Theta^{Aa}(\sigma,\tau)$ with $A = 1,...,\mathcal{N}$ where $\mathcal{N}$ denotes the number of supersymmetries and $a = 1,...,2^{D/2}$ is the spinor index. The infinitesimal supersymmetry transformations are then

$$\delta\Theta^{Aa} = \epsilon^{Aa} \quad , \quad \delta X^\mu = \bar{\epsilon}^A \Gamma^\mu \Theta^A \tag{2.12}$$

where $\Gamma^\mu$ are $D$-dimensional Dirac matrices. A convenient starting point is to construct two theories with closed strings, namely Type IIA and Type IIB string theory. According to the fact that closed strings have left- and right-movers, these theories have $\mathcal{N} = 2$ and therefore two fermionic coordinates $\Theta^{1,2}$. If we define the supersymmetric combination

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A \tag{2.13}$$

then the Green-Schwarz action (with $\alpha' = 1$) is written as

$$\begin{aligned} S = -\frac{1}{2\pi}\int d^2\sigma \Big(\sqrt{-h}h^{\alpha\beta}\Pi_\alpha \cdot \Pi_\beta - 2\epsilon^{\alpha\beta}\big[ - \partial_\alpha X^\mu(\bar{\Theta}^1\Gamma_\mu\partial_\beta\Theta^1 - \bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2) \\ - \bar{\Theta}^1\Gamma_\mu\partial_\alpha\Theta^1\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2\big]\Big) \end{aligned} \tag{2.14}$$

Remarkably, the $\Theta^A$ satisfy the equations

$$(\partial_\tau + \partial_\sigma)\Theta^1 = 0\,, \quad (\partial_\tau - \partial_\sigma)\Theta^2 = 0 \tag{2.15}$$

and therefore describe waves propagating in opposite directions along the string world-sheet. We can then distinguish Type IIA and IIB theories by the chirality of $\Theta^{1,2}$:

$$\text{IIA:} \quad \Gamma_{11}\Theta^A = (-1)^{(A+1)}\Theta^A\,, \qquad \text{IIB:} \quad \Gamma_{11}\Theta^A = \Theta^A \tag{2.16}$$

Both of the spinors a priori have $2^{D/2}$ complex components, which in our case $D = 10$ gives 32 components. By imposing Majorana and Weyl conditions, we get 16 real components. Additionally, a local symmetry called Kappa symmetry shows that half of the degrees of freedom of $\Theta^A$ can be decoupled and are non-propagating, i.e. gauge degrees of freedom. Therefore $\Theta^1, \Theta^2$ have 8 real degrees of freedom each. In light-cone gauge we choose

$$\Gamma^+\Theta^A = 0\,, \quad \Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9)\,, \quad X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^9) \tag{2.17}$$

Note that only the eight transverse coordinates of $X^\mu$ are propagating degrees of freedom in this gauge. Thus the ten-dimensional Lorentz invariance under $SO(9,1)$ transformations is reduced to a transverse $SO(8)$ symmetry, which means that the $X^\mu$ give the fundamental vector representation of $Spin(8)$, denoted by $\mathbf{8_v}$. The components of the $\Theta^A$ that are not gauged away form distinct spinor representations of $Spin(8)$, which is the double cover of $SO(8)$. The spinor representations $\mathbf{8_s}, \mathbf{8_c}$ of $Spin(8)$ describe spinors of opposite chirality and are related, together with $\mathbf{8_v}$, by a $S_3$-symmetry called triality.

The two representations combine as $\mathbf{8_s} \oplus \mathbf{8_c}$ for Type IIA (opposite chirality) and $\mathbf{8_s} \oplus \mathbf{8_s}$ for Type IIB (same chirality). We denote the spinor components remaining after gauge fixing by $S_i^a$ for $\mathbf{8_s}$ and $S_i^{\dot{a}}$ for $\mathbf{8_c}$. The action in light-cone gauge in compact form is then

$$S = -\frac{1}{2\pi}\int d^2\sigma\, \partial_\alpha X_i \partial^\alpha X^i \qquad\qquad i = 1, ...8$$

$$\left( + \frac{i}{\pi}\int d^2\sigma\, (S_1^a \partial_+ S_1^a + S_2^{\dot{a}} \partial_- S_2^{\dot{a}}) \right) \qquad \text{(Type IIA)} \tag{2.18}$$

$$\left( + \frac{i}{\pi}\int d^2\sigma\, (S_1^a \partial_+ S_1^a + S_2^a \partial_- S_2^a) \right) \qquad \text{(Type IIB)} \tag{2.19}$$

The equations of motion then take the form

$$\partial_+\partial_- X^i = 0\,, \quad \partial_+ S_1^a = 0\,; \qquad \text{IIA: } \partial_- S_2^{\dot{a}} = 0 \text{ / IIB: } \partial_- S_2^a = 0 \tag{2.20}$$

The spinors are again expanded in their Fourier modes and anti-commutation relations are imposed to quantize the theory. The result obtained earlier in the RNS formalism is now given in a single sector and without the need for the GSO-projection, as there is no tachyon in the spectrum.

Since the left- and right-moving spinors in Type IIB theory are in the same representation $\mathbf{8_s}$ or $\mathbf{8_c}$, we can use an orientifold projection to relate Type IIB to a theory of open strings called Type I string theory [10, 11], which reduces the number of supersymmetries to $\mathcal{N} = 1$.

The spectrum of the ground state of the open string is described by the 16-dimensional multiplet $\mathbf{8_v} \oplus \mathbf{8_{c,s}}$. For closed strings we form tensor products of these multiplets, giving $16 \times 16 = 256$ states. We get the following multiplets:

$$\text{Type IIA: } (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \,, \qquad \text{Type IIB: } (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \tag{2.21}$$

The individual tensor products for the bosonic states are

$$\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} + \mathbf{28} + \mathbf{35} \qquad \text{(Common sector)} \tag{2.22}$$

$$\mathbf{8_c} \otimes \mathbf{8_s} = \mathbf{8_v} + \mathbf{56_t} \qquad \text{(Type IIA)} \tag{2.23}$$

$$\mathbf{8_c} \otimes \mathbf{8_c} = \mathbf{1} + \mathbf{28} + \mathbf{35}_+ \qquad \text{(Type IIB)} \tag{2.24}$$

where $\mathbf{56_t}$ corresponds to a three-form tensor field and $\mathbf{35}_+$ describes a four-form self-dual tensor field. This is the same spectrum that is obtained in the RNS formalism. The first sector (corresponding to the NS-NS sector) contains the same fields for Type IIA and IIB:

$$\mathbf{8_v} \otimes \mathbf{8_v} \;\equiv\; \phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} \tag{2.25}$$

## 2.3. Supergravity and low-energy effective actions

### 2.3.1. Motivation

When considering the low energy limit of a string theory in a Minkowski background, we observe that for large string tension $T \to \infty$, or weak coupling $g_s \to 0$, the masses of the massive states become large compared to the massless states. The theory can then be approximated by a 10-dimensional supergravity theory that only contains the massless fields and interactions. Further, it was shown that Type IIA string theory is the low energy limit of 11-dimensional supergravity [12]. 11D supergravity [13] has been a promising candidate for a grand unified theory, dating back to the discovery that $D = 11$ is the highest possible dimension for a supergravity theory containing a single spin-2 graviton field [14], and has been conjectured to be itself the low-energy limit of a more fundamental theory called M-Theory [15, 16].

### 2.3.2. Bosonic actions

We now list some relevant low energy effective actions for string theories [8]. First, the bosonic string action with massless fields $\phi, g_{\mu\nu}$ and $B_{\mu\nu}$ is written as

$$S_B = \frac{1}{2\kappa_{26}^2} \int d^{26}x \sqrt{-G}\, e^{-2\phi} \Big( R + 4(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}|H_3|^2 \Big) \tag{2.26}$$

where $R$ is the curvature scalar, $\kappa_{26}$ is the 26-dimensional gravitational coupling constant and $H_3 = dB_2$ such that $B_2$ is the two-form gauge field with components $B_{\mu\nu}$, explicitly

$$B_2 \equiv B = \frac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu . \tag{2.27}$$

Note the usual convention to write $p$-dimensional antisymmetric tensor fields as $A_p$, etc.

The bosonic action of 11-dimensional supergravity is written as

$$2\kappa_{11}^2 S = \int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4 \tag{2.28}$$

where $F_4 = dA_3$ is a four-form field strength associated with the three-form gauge field $A_3$. By dimensional reduction on a circle, one can derive the action of 10-dimensional Type IIA supergravity, which can be written as $S_{IIA} = S_{NS} + S_R + S_{CS}$ where $S_{NS}$ contains the dynamical terms involving the fields from the NS-NS sector, $S_R$ contains the Ramond fields and $S_{CS}$ is a topological Chern-Simons term which is independent of the metric [17].

In the string frame, which is characterized by an exponential term dependent on the dilaton, the action for the NS-fields is written as

$$S_{NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\,e^{-2\phi}\Big(R + 4(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}|H_3|^2\Big) \tag{2.29}$$

where $g$ is the ten-dimensional metric, $\phi$ is the dilaton field and $H_3 = dB_2$ is the three-form field strength associated to the two-form potential $B_2$. For the R-R fields we get

$$S_R = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(|F_2|^2 + |\tilde{F}_4|^2\right) \tag{2.30}$$

where $F_2 = dC_1$ is the two-form field strength associated to the one-form gauge field $C_1$ and the rescaled tensor field

$$\tilde{F}_4 = dC_3 + C_1 \wedge H_3 \tag{2.31}$$

is the four-form field strength associated to the three-form $C_3$ with a correction that is needed for gauge invariance. The Chern-Simons term reads

$$S_{CS} = -\frac{1}{4\kappa_{10}^2}\int B_2 \wedge F_4 \wedge F_4 \tag{2.32}$$

For the Type IIB theory, we have:

$$S_R = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) \tag{2.33}$$

$$S_{CS} = -\frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3 \tag{2.34}$$

$$F_{n+1} = dC_n, \quad H_3 = dB_2 \tag{2.35}$$

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3 \tag{2.36}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \tag{2.37}$$

Note that the Type IIB case also has a self-duality condition that has to be imposed on the equations of motion as a constraint:

$$\tilde{F}_5 = \star\tilde{F}_5 \tag{2.38}$$

## 2.4. Dimensional Reduction & Compactification

### 2.4.1. Decomposition of representations

In order to match predictions from a 10-dimensional superstring theory on a higher-dimensional manifold $\mathcal{M}$ to our observed spacetime with $D = 4$, the concept of compactification is introduced by decomposing the target manifold $\mathcal{M}$ as a product space $M_4 \times K_6$ where $M_4$ is a four-dimensional Lorentzian manifold and $K_6$ is a six-dimensional compact manifold [18]. In particular, when $K_6$ is a Calabi-Yau threefold, it is possible to obtain a four-dimensional theory on $M_4$ with minimal unbroken supersymmetry $\mathcal{N} = 1$.

To understand how representations of the Lorentz group decompose under compactifications, consider a vector field $A_M$ transforming in the fundamental representation $\mathbf{D}$ of $SO(1, D-1)$. Corresponding to the split of the manifold into $M_d \times K_{D-d}$, the Lorentz group is split into $SO(1, d-1) \times SO(D-d)$. Thus, the representation $\mathbf{D}$ is split as $(\mathbf{d}, \mathbf{1}) + (\mathbf{1}, \mathbf{D} - \mathbf{d})$, giving a vector $A_\mu$ transforming under $SO(1, d-1)$ and $(D-d)$ scalars transforming under $SO(D-d)$. Similarly, an antisymmetric tensor field $B_{MN}$ decomposes into $d$-dimensional antisymmetric tensor $B_{\mu\nu}$, vectors $B_{\mu m}$ and scalars $B_{mn}$ [18].

As discussed in previous sections, supersymmetric theories have a set of conserved spinor charges $Q^A$, $A = 1, ..., \mathcal{N}$ with $2^{D/2}$ components. For example, the decomposition of the Weyl representation $\mathbf{16}$ of $SO(1, 9)$ into $SO(1, 3) \times SO(6)$ yields

$$\mathbf{16} = (\mathbf{2_L}, \bar{\mathbf{4}}) \oplus (\mathbf{2_R}, \mathbf{4}) \tag{2.39}$$

where $\mathbf{4}, \bar{\mathbf{4}}$ are Weyl spinors transforming in $SO(6) \simeq SU(4)$ and $\mathbf{2_L}, \mathbf{2_R}$ are chiral Weyl spinors that are conjugate to each other in $SO(1, 3)$. In $D = 10$ the Majorana condition can be imposed on spinors along with the Weyl condition, giving four Majorana spinors in $D = 4$. Therefore, we get supersymmetry $\mathcal{N} = 4, 8$ in $D = 4$ from $\mathcal{N} = 1, 2$ (Type I/Heterotic, Type II) string theory in $D = 10$ by dimensional reduction on a torus $T^6$.

On compact manifolds with special holonomy, meaning manifolds with a holonomy group that is a subgroup of the generic $SO(n)$ holonomy group of orientable manifolds, the number of supersymmetries can be reduced. For example, if we compactify $\mathcal{N} = 1, 2$ in $D = 10$ on a six-dimensional manifold with holonomy group $SU(2)$, the number of supersymmetries is $\mathcal{N} = 2, 4$. For $SU(3)$, this number is reduced further to $\mathcal{N} = 1, 2$; e.g. Type II with $\mathcal{N} = 2$ on a $SU(3)$ manifold has $\mathcal{N} = 2$ in $D = 4$. This is useful to construct minimally supersymmetric extensions of Standard Model theories from string theory.

### 2.4.2. Kaluza-Klein reductions

The simplest possible way of reducing the dimensionality of a given theory is by decomposing the original spacetime as $\mathcal{M}_{\mathcal{D}} = M_{D-1} \times S_1$, as pioneered by Kaluza and Klein [19, 20]. In contrast to a general compactification, here often only the zero Fourier modes of various fields are considered. As an example, we consider the dimensional reduction of

$D = 11$ supergravity to Type IIA supergravity in $D = 10$, which was mentioned earlier in section 2.3.2. Recall that the bosonic field content of $D = 11$ supergravity is given by the metric $G$ and a three-form gauge field $A_3$. The metric is decomposed as

$$G_{MN} = e^{-2\phi/3} \begin{pmatrix} g_{\mu\nu} + e^{2\phi} A_\mu A_\nu & e^{2\phi} A_\mu \\ e^{2\phi} A_\nu & e^{2\phi} \end{pmatrix} \tag{2.40}$$

where $g_{\mu\nu}$ is the ten-dimensional metric, $A_\mu$ is a $U(1)$ gauge field and $\phi$ is a dilaton field. The three-form field is split in $D = 10$ into a new three-form with components $A_{\mu\nu\rho} = A^{(11)}_{\mu\nu\rho}$ and a two-form $B_{\mu\nu} = A^{(11)}_{\mu\nu,11}$. The corresponding field strengths are given by $F^{(11)}_{\mu\nu\rho\lambda} = F_{\mu\nu\rho\lambda}$ and $F^{(11)}_{\mu\nu\rho,11} = H_{\mu\nu\rho}$. Considering the form of the metric, these can be decomposed as:

$$F^{(11)}_{abcd} = e^{4\phi/3}(F_{abcd} + 4A_{[a}H_{bcd]}) = e^{4\phi/3}\tilde{F}_{abcd}\,, \qquad F^{(11)}_{abc,11} = e^{\phi/3}H_{abc} \tag{2.41}$$

We have therefore obtained the rescaled tensor field (2.31), and together with the remaining field content, the action of bosonic Type IIA supergravity in $D = 10$ is constructed.

### 2.4.3. Compactification on Calabi-Yau manifolds

As mentioned earlier, when compactifying string theory on a manifold with special holonomy where the holonomy group is $SU(n)$, we can break some of the supersymmetry. Complex manifolds with $SU(n)$ that are suitable for compactifications are the Calabi-Yau manifolds, which are introduced briefly in Appendix A.

In addition to the $SU(n)$ holonomy, Calabi-Yau manifolds have a Ricci-flat metric that admits covariantly constant spinor fields, also called Killing spinors, such that $\nabla_\mu \epsilon = 0$. The elements of the holonomy group act on spinors under parallel transport around a loop on the compact manifold, but a covariantly constant spinor is unchanged under elements of the holonomy group $\mathcal{H}$ - meaning $\epsilon$ must be a singlet $\mathbf{1}$ under $\mathcal{H}$. The Weyl spinor $\mathbf{4}$ from (2.39) in $SO(6) \simeq SU(4)$ decomposes under non-trivial holonomy groups as [18]:

$$\mathbf{4}_{\mathbf{SU(4)}} = (\mathbf{3} \oplus \mathbf{1})_{\mathbf{SU(3)}} = (\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1})_{\mathbf{SU(2)}} \tag{2.42}$$

Combined with an analogous decomposition for $\bar{\mathbf{4}}$, we therefore have two covariantly constant spinors $\epsilon_\pm$ under $\mathcal{H} = SU(3)$ (left- and right-handed), while under $SU(2)$ we have two such spinors of each chirality. For the torus $T^6$ with trivial $\mathcal{H}$ we then have four covariantly constant spinors $\epsilon^i_\pm$. For example, consider the compactification to $D = 4$ under the decomposition (2.39). If we take the internal manifold to be a Calabi-Yau threefold, which has the holonomy group $SU(3)$, we can write the supersymmetry parameter $\epsilon$ as:

$$\epsilon = \epsilon_L \otimes \epsilon_- + \epsilon_R \otimes \epsilon_+ \tag{2.43}$$

From the Majorana condition on $\epsilon$ we have $\epsilon_L^* = \epsilon_R$, giving a single Majorana spinor corresponding to the $SO(1,3)$ group of the decomposition $SO(1,3) \times SO(6)$. Therefore we have shown that $D = 10$, $\mathcal{N} = 1$ string theory compactified to $D = 4$ on a Calabi-Yau threefold gives again $\mathcal{N} = 1$. Similarly, we can compactify on the product space $K3 \times T^2$, where the $SU(2)$-holonomy of the Calabi-Yau twofold $K3$ gives us $\mathcal{N} = 2$ in $D = 4$.

# 3. BPS States, D-Branes and String Duality

## 3.1. Supersymmetry algebras and BPS bound

We now look at the algebraic structure of supersymmetry [21, 22]. The Poincaré algebra is extended by supersymmetry to the super-Poincaré algebra. Recall the algebra of the generators $P^\mu, M^{\mu\nu}$ corresponding to translations and Lorentz transformations:

$$[P^\mu, P^\nu] = 0 \tag{3.1}$$
$$[M^{\mu\nu}, P^\sigma] = i\,(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}) \tag{3.2}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}) \tag{3.3}$$

First consider the case $D = 4$. We include spinor generators $Q^A_\alpha$, $\bar{Q}^A_{\dot{\alpha}}$ from the fundamental and conjugate representations of $SL(2,\mathbb{C})$, with $A = 1, ..., \mathcal{N}$ where $\mathcal{N}$ denotes the number of supersymmetries and $\alpha = 1, ..., 4$ denotes the spinor component. For $\mathcal{N} = 1$ we have *minimal* supersymmetry, while the case $\mathcal{N} > 1$ is called *extended* supersymmetry. The additional commutation relations for the minimal supersymmetry algebra $\mathcal{N} = 1$ are:

$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta\,, \qquad [Q_\alpha, P^\mu] = [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0 \tag{3.4}$$
$$\{Q_\alpha,\, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}},\, \bar{Q}_{\dot{\beta}}\} = 0\,, \qquad \{Q_\alpha,\, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \tag{3.5}$$

with Pauli matrices $\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$, $\sigma^0 = \mathbb{1}$ and $\sigma^{\mu\nu}$ are antisymmetrized products:

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}[\sigma^\mu,\, \bar{\sigma}^\nu]_\alpha{}^\beta \tag{3.6}$$

We can extend the supersymmetry algebra to a number of supersymmetries $\mathcal{N}$ introducing additional spinor generators with indices $A = 1, ..., \mathcal{N}$. The algebra for $Q^A_\alpha$, $\bar{Q}^A_{\dot{\alpha}}$ is then:

$$\begin{aligned} \{Q^A_\alpha,\, Q^B_\beta\} &= \epsilon_{\alpha\beta} Z^{AB} \\ \{Q^A_\alpha,\, \bar{Q}^B_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu\, \delta^{AB} \end{aligned} \tag{3.7}$$

where $Z^{AB}$ is the antisymmetric central charge matrix that commutes with all generators.

### 3.1.1. Supersymmetry algebra in higher dimensions

The Dirac matrices in $D$ dimensions are representations of the Clifford algebra:

$$\{\Gamma^M,\, \Gamma^N\} = 2\eta^{MN} \tag{3.8}$$

Then the antisymmetric objects $\Sigma^{MN}$ are the generators of the Lorentz group $SO(1, D{-}1)$:

$$\Sigma^{MN} = \frac{i}{4}[\Gamma^M,\, \Gamma^N] \tag{3.9}$$

The chirality operator in even dimensions $D = 2n$ is defined as $\Gamma^{2n+1} = i^{n-1}\,\Gamma^0\,\Gamma^1 ... \Gamma^{2n-1}$ and the supersymmetry anticommutation relations have the general form:

$$\{Q_\alpha,\, Q_\beta\} = a^M_{\alpha\beta} P_M + Z_{\alpha\beta} \tag{3.10}$$

### 3.1.2. Massive representations and the BPS bound

For $D = 4$ and extended supersymmetry $\mathcal{N} > 1$ we consider a massive representation such that $p^\mu = (M, 0, 0, 0)$. The anticommutator for $Q^A_\alpha$, $\bar{Q}^B_{\dot{\beta}}$ then becomes:

$$\{Q^A_\alpha, \bar{Q}^B_{\dot{\beta}}\} = 2M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta^{AB} \tag{3.11}$$

For central charges $Z^{AB} \neq 0$ the BPS bound for massive states with mass $M$ is then [21]:

$$M \geq \frac{1}{2\mathcal{N}} Tr\{\sqrt{Z^\dagger Z}\} \tag{3.12}$$

States which saturate this bound are called BPS states (after Bogomol'nyi, Prasad and Sommerfield [23, 24]). For example, take $\mathcal{N} = 2$ with the following central charge matrix:

$$Z^{AB} = \begin{pmatrix} 0 & Z_1 \\ -Z_1 & 0 \end{pmatrix} \tag{3.13}$$

This gives the BPS condition

$$M \geq \frac{|Z_1|}{2} \,. \tag{3.14}$$

Generally, for an even number of supersymmetry with $\mathcal{N} > 2$, the central charge matrix takes a block-diagonal form:

$$Z^{AB} = \begin{pmatrix} 0 & Z_1 & 0 & 0 & \\ -Z_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & Z_2 & \\ 0 & 0 & -Z_2 & 0 & \\ & \vdots & & & \ddots \end{pmatrix} \tag{3.15}$$

The BPS bound then holds for each block:

$$M \geq \frac{|Z_1|}{2}, \, M \geq \frac{|Z_2|}{2}, \, \dots \tag{3.16}$$

We can introduce a classification of BPS states given by the amount of BPS conditions that are saturated. For example, in the case $\mathcal{N} = 4$, we have two central charges $Z_1, Z_2$ and the BPS conditions are the first two listed above. States that saturate both equations are called *half*-BPS, meaning that they leave half of the supersymmetries unbroken, while states which saturate only one of the two bounds are called *quarter*-BPS, in which case they only leave a quarter of the supersymmetries unbroken. An important conclusion is that BPS states are stable, because the BPS condition means that the mass is tied to a central charge. As long as the BPS multiplet is not degenerate with another representation, it is therefore possible to extrapolate predicitions from weak to strong coupling in the presence of BPS states [8]. Furthermore, in Type II string theories half-BPS states can be realized as stable D-branes, which will be discussed in the following chapter.

## 3.2. D-Branes & Ramond-Ramond charges

D-branes are extended objects in Type II string theories that carry Ramond-Ramond charges and break half of the supersymmetries of the string [25, 26]. D-branes are hypersurfaces that admit open strings with mixed boundary conditions on the coordinates, confining the endpoints of the strings to live on the D-brane. The boundary conditions of a D$p$-brane are Neumann on $p+1$ coordinates and Dirichlet on the other $9-p$ coordinates:

$$n^\alpha \partial_\alpha X^\mu = 0, \quad \mu = 0, ..., p \tag{3.17}$$

$$X^m = 0, \quad m = p+1, ..., 9 \tag{3.18}$$

An important consequence of this is that the ten-dimensional Lorentz group $SO(1,9)$ is broken to $SO(1,p) \times SO(9-p)$. Considering the supersymmetries, we start from a Type II theory with 32 conserved supercharges $Q_\alpha, \tilde{Q}_\alpha$ (which are Majorana-Weyl spinors with 16 real components) corresponding to $\mathcal{N} = 2$. In the presence of a D-brane, only the linear combination of left- and right-moving supercharges $Q_\alpha + \Gamma\tilde{Q}_\alpha$ is conserved - closed strings from the bulk of the spacetime couple to open strings on the D-brane. Therefore, the supersymmetries are reduced to $\mathcal{N} = 1$, and so the D-brane is a half-BPS state [8].

Furthermore, the world volume of a D$p$-brane couples to $(p+1)$-form potential $A_{(p+1)}$ with field strength $F_{(p+2)}$ - it acts as a RR-charged solitonic state. Note that the spectrum of open strings on a D-brane is also tachyon-free. Looking at the Type IIA and IIB Ramond fields and their duals, we get the following stable D$p$-branes:

- Type IIA: $p = 0, 2, 4, 6, 8$ ($p = 0$: point particle)
- Type IIB: $p = -1, 1, 3, 5, 7, 9$ ($p = -1$: Dirichlet instanton)

The D-brane couples to the fields via the following action:

$$S_{int} = i\mu_p \int_{branes} A_{p+1} = \frac{i\mu_p}{(p+1)!} \int d^{p+1}\sigma A_{\mu_1 ... \mu_{p+1}} \frac{\partial x^{\mu_1}}{\partial \sigma^0} \cdots \frac{\partial x^{\mu_{p+1}}}{\partial \sigma^p} \tag{3.19}$$

For the complete action of the $(p+1)$-form potential, we include the kinetic term from the closed string sector discussed above in section 2.3.2:

$$S = S_{dyn} + S_{int} = \frac{1}{2} \int F_{p+2} \wedge \star F_{p+2} + i\mu_p \int_{branes} A_{p+1} \tag{3.20}$$

To introduce a way of obtaining charges from fields, we first consider Maxwell's equations in four dimensions with electric and magnetic sources given by current 1-forms $J = J_\mu dx^\mu$:

$$dF = \star J_m \,, \quad d \star F = \star J_e \tag{3.21}$$

The differential form of Gauss' law and the magnetic flux integral then relate the charges to integrals of the field strengths over a two-sphere around the charge:

$$e = \int_{S^2} \star F \,, \qquad g = \int_{S^2} F \tag{3.22}$$

These charges must satisfy the Dirac quantization condition:

$$e \cdot g \in 2\pi\mathbb{Z} \tag{3.23}$$

For D$p$-branes, Gauss' law and the magnetic counterpart are:

$$\mu_p = \int \star F_{p+2}\,, \qquad \mu_{6-p} = \int F_{p+2} \tag{3.24}$$

Note that since $\star F_{p+2}$ is a $(D-(p+2))$-form, $\int \star F_{p+2}$ is an integral over $S^{D-p-2}$, which is a sphere of the dimension required to surround a $p$-brane. The magnetic flux integral $\int F_{p+2}$ is integrated over $S^{p+2}$, which is a sphere that can surround a $(D-p-4)$-brane. This suggests that the magnetic dual of a $p$-brane in $D = 10$ string theory is a $(6-p)$-brane. An important result is that the charges of these objects also satisfy a Dirac quantization:

$$\mu_p \cdot \mu_{6-p} \in 2\pi\mathbb{Z} \tag{3.25}$$

## 3.3. Kaluza-Klein monopoles

Another simple kind of solitonic BPS state is the Kaluza-Klein monopole [27], which is a solution that arises from Kaluza-Klein theory as already mentioned in section 2.4.2. Consider a theory of gravity with the Einstein-Hilbert action in $D = 5$:

$$S = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^5x\sqrt{-g}\,R \tag{3.26}$$

The manifold $\mathcal{M}$ is assumed to be $M_4 \times S^1$ where the circle has radius $R$. The expansion of the metric in a Fourier series is:

$$g_{AB}(x^\mu, x^5) = \sum_{n=-\infty}^{+\infty} g_{AB}(x^\mu)e^{inx^5/R} \tag{3.27}$$

In the low energy limit we only consider the zero mode $n = 0$, which lets us take the metric to be independent of $x^5$. The line-element and the metric are written as

$$ds^2 = V(dx^5 + A_\mu dx^\mu)^2 + g_{\mu\nu}dx^\mu dx^\nu \tag{3.28}$$

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu V & A_\mu V \\ A_\nu V & V \end{pmatrix} \tag{3.29}$$

where $A_\mu$ is a vector gauge field that arises from the local $U(1)$ gauge symmetry of $x^5$ and $V$ is a scalar pseudo-Goldstone boson that appears due to spontaneous breaking of global scale invariance by fixing the radius $R$. This gives us an effective action in $D = 4$:

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g^4}\,V^{1/2}(R_4 + \frac{1}{4}VF_{\mu\nu}F^{\mu\nu}) \tag{3.30}$$

The Kaluza-Klein monopole is a static, Ricci-flat soliton given by the Taub-NUT metric:

$$ds^2 = -dt^2 + \left(1+\frac{4m}{r}\right)^{-1}\left(dx^5 + A_\phi\, d\phi\right)^2 + \left(1+\frac{4m}{r}\right)\left(dr^2 + r^2 d\Omega^2_{S^2}\right) \tag{3.31}$$

$$A_\phi = 4m(1-\cos\theta) \quad \rightarrow \quad \vec{B} = \boldsymbol{\nabla}\times\vec{A} = \frac{4m\vec{r}}{r^3} \tag{3.32}$$

The identification $16\pi m \sim 2\pi R$ is taken so that the solution is asymptotically flat and, to make the metric singularity-free, $x^5$ is taken to have periodicity $16\pi m$. In Type II string theory, the KK-monopole breaks half of the supersymmetry, making it a half-BPS state.

## 3.4. String duality

In this section we will summarize some basic aspects of string dualities. There are five different constructions of the superstring, namely: Type I, Type IIA and IIB, heterotic with gauge group $SO(32)$ and heterotic with gauge group $E_8 \times E_8$. Furthermore, $D = 11$ supergravity and M-theory have been considered as candidates for a theory of everything. Surprisingly, these theories have found to be connected via different kinds of dualities, for example $D = 11$ SUGRA $\longleftrightarrow$ Type IIA (reduction on $S^1$) and Type I $\longleftrightarrow$ Heterotic $SO(32)$ (S-duality). There are also string-string dualities between string theories of the same category, between heterotic $SO(32)$ and $E_8 \times E_8$ theory via T-duality, or between type II theories. For a thorough discussion of duality in string theory we refer to [12, 28].

For our purposes, the duality between Type II string theories compactified on $K3 \times S^1$ and heterotic string theory compactified on $T^5$ is of interest, combined with the observation that Type IIA and Type IIB are related by T-duality in $D < 10$.

**The heterotic string in $D = 10$**

As mentioned before, the heterotic string can be formulated with one of two gauge groups (indicated by anomaly cancellation): $SO(32)$ and $E_8 \times E_8$. The latter is especially intriguing, due to a chain of embeddings containing the gauge group of the standard model:

$$SU(3) \times SU(2) \times U(1) \subset SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8 \qquad (3.33)$$

In the bosonic construction of the heterotic string, the left- and right-moving sectors corresponds to bosonic string theory and superstring theory, respectively. In light-cone gauge with the RNS formalism, we have the following worldsheet fields:

| | | |
|---|---|---|
| Left-moving bosonic string: | $X^i(\sigma^+),\ X^I(\sigma^+)$ | $I = 1, ..., 16$ |
| Right-moving superstring: | $X^i(\sigma^-),\ \tilde{\psi}^i(\sigma^-)$ | $i = 1, ..., 8$ |

Here the $X^i$ are the usual bosonic and transverse string coordinates, $\tilde{\psi}^i$ are worldsheet fermions and $X^I$ are coordinates of an internal $E_8 \times E_8$ torus. The corresponding oscillator modes are $\alpha_n^i, \tilde{\lambda}_n^i$ and $\beta_n^I$. Recall that the Ramond and Neveu-Schwarz sectors have periodic and anti-periodic boundary conditions on the fermionic coordinates. For this discussion, we focus on the periodic Ramond sector. The Virasoro operators of left- and right-movers give the corresponding oscillator numbers $N$ and $\tilde{N}$:

$$L_0 = \sum_{n=1}^{\infty} \Big( \sum_{i=1}^{8} \alpha_{-n}^i \alpha_n^i + \sum_{i=1}^{16} \beta_{-n}^i \beta_n^i \Big) - 1 := N - 1 \qquad \text{(Bosonic)} \qquad (3.34)$$

$$\tilde{L}_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{8} \Big( \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \tilde{\lambda}_{-r}^i \tilde{\lambda}_r^i \Big) - \frac{1}{2} := \tilde{N} - \frac{1}{2},\ r = -(n - \frac{1}{2}) \qquad \text{(NS-sector)} \qquad (3.35)$$

$$\tilde{L}_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{8} \Big( \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \tilde{\lambda}_{-r}^i \tilde{\lambda}_r^i \Big) := \tilde{N}, \qquad r = (n - 1) \qquad \text{(R-sector)} \qquad (3.36)$$

# 4. Microstate Counting in String Theory

## 4.1. Heterotic strings and Dabholkar-Harvey states

As a simple example of black hole states arising from string theory, we look at a construction of the heterotic string on a torus. As we will show, we can get BPS states without introducing solitonic D-branes, which are not present in heterotic string theory as it does not contain open strings and RR-charged fields. This is achieved by considering a spectrum consisting of an infinite tower of massive states. The physical implication is that, for large enough coupling constants and/or sufficiently massive excitations, the Hilbert space of string excitations contains black hole states in the noncompact dimensions [8].

We start with the heterotic string and compactify on a torus $T^6$ to a target space with $D = 4$. This gives 28 $U(1)$ gauge fields, which transform as a vector under the T-duality group $O(22, 6; \mathbb{Z})$. As a result of compactification, allowed charges are confined to an even self-dual lattice, which is called the Narain lattice. The charges are encoded in the internal momenta $p_L$ (22 components) and $p_R$ (6 components), which are related to the total oscillator number by:

$$p_R^2 - p_L^2 = 2N_{total} := \frac{Q^2}{2} \tag{4.1}$$

First, we set up a perturbative string state by wrapping around a $S^1$ with radius $R$ where the string has winding number $w$ and quantized momentum $n$. Half-BPS states in this construction are obtained by keeping the right-movers in the ground state. The left- and right-moving momenta for $\alpha' = 1$ are [4]:

$$p_{L,R} = \frac{1}{\sqrt{2}}\Big(\frac{n}{R} \pm wR\Big) \tag{4.2}$$

The Virasoro constraints for this system are:

$$L_0 - \frac{M^2}{4} + \frac{p_L^2}{2} = 0 \tag{4.3}$$

$$\tilde{L}_0 - \frac{M^2}{4} + \frac{p_R^2}{2} = 0 \tag{4.4}$$

With $\tilde{N} = 0$ we then have a BPS bound, relating $p_R$ to the mass and further to the central charge of the supersymmetry algebra:

$$M = \sqrt{2} p_R \tag{4.5}$$

Since $N$ is still arbitrary, we get a tower of stable states, which are called *Dabholkar-Harvey* states. From the Virasoro constraint for $L_0$, we get:

$$N_{total} = N - 1 = \frac{1}{2}(p_R^2 - p_L^2) = nw \tag{4.6}$$

To compute the degeneracy of such states for any given $n, w$, we evaluate the canonical partition function given by

$$Z(\beta) = Tr(q^{L_0}\bar{q}^{\tilde{L}_0}) = Tr(e^{-\beta L_0}) \equiv \sum_{N=-1}^{\infty} d(N)q^N \tag{4.7}$$

with the micro-canonical degeneracy $d(N)$ and the fugacity $q$, which is defined as

$$q = e^{-\beta} = e^{2\pi i\tau} \tag{4.8}$$

The degeneracy $d(N)$ counts the ways in which 24 left-moving massless bosons in 1+1 dimensions give $N$ number of excitations at the temperature $1/\beta$. The partition function can be written as:

$$Z(\beta) = \sum_{N=-1}^{\infty} d(N)q^N = \frac{1}{\Delta(q)} \tag{4.9}$$

The discriminant function $\Delta(q)$ is related to the Dedekind $\eta$-function by

$$\Delta(q) = \eta(q)^{24} = q\prod_{s=1}^{\infty}(1-q^s)^{24} \tag{4.10}$$

This allows us to write the partition function as:

$$Z(\beta) = \frac{1}{q}\prod_{s=1}^{\infty}\frac{1}{(1-q^s)^{24}} \tag{4.11}$$

This gives exactly the partition function of 24 free bosons corresponding to the bosonic degrees of freedom of the heterotic string in light-cone gauge. For comparison, the partition function of a single free left-moving boson in a $(1+1)$-dimensional conformal field theory on a torus gives [29]:

$$Z_{bos}(q) = Tr(q^{L_0}) = q^{-\frac{1}{24}}\prod_{s=1}^{\infty}\frac{1}{1-q^s} \tag{4.12}$$

Now we make use of the modular properties of $\eta(\tau)$. The modular group is the projective special linear group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ that contains the modular transformations:

$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad \text{where } a,b,c,d \in \mathbb{Z} \quad \text{and } (ad-bc)=1 \tag{4.13}$$

The Dedekind $\eta$-function $\eta(\tau)$ transforms as

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau) \tag{4.14}$$

and the discriminant function $\Delta(\tau)$ transforms a modular form of weight 12:

$$\Delta(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{12}\Delta(\tau) \tag{4.15}$$

Since the partition function is given by $Z(\beta) = 1/\Delta(\beta)$, it is also a modular form. With our definition of $q = e^{-\beta} = e^{2\pi i\tau}$ we then have:

$$-i\tau = \beta/2\pi\,, \qquad -1/\tau = 2\pi i/\beta \tag{4.16}$$

Inserting $-1/\tau$, we then have $q = e^{(2\pi i)^2/\beta} = e^{-4\pi^2/\beta}$. Defining the new argument as $\tilde{\beta} = 4\pi^2/\beta$ with $\tilde{q} = e^{-\tilde{\beta}}$, the partition function now reads:

$$Z(\beta) = \Big(\frac{\beta}{2\pi}\Big)^{12} Z(\tilde{\beta})\,, \quad Z(\tilde{\beta}) = Z\Big(\frac{4\pi^2}{\beta}\Big) \tag{4.17}$$

Note that the high temperature limit $\beta \to 0$ is $q \to 1$, while the low temperature limit $\beta \to \infty$ is $q \to 0$. We look at the asymptotic behavior of the partition function by taking $\beta \to 0$ and thus $\tilde{\beta} \to \infty$. By the definition of $\tilde{q} = e^{-\tilde{\beta}}$ we have $\tilde{q} \to 0$ in this limit. We can now approximate the partition function as follows:

$$Z(\tilde{\beta}) = \frac{1}{\tilde{q}} \prod_{s=1}^{\infty} \frac{1}{(1-\tilde{q}^s)^{24}} \sim \frac{1}{\tilde{q}} \tag{4.18}$$

If we compute the degeneracy $d(N)$, we get an inverse Laplace transform:

$$d(N) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\epsilon - iT}^{\epsilon + iT} d\beta\, e^{\beta N} Z(\beta) \tag{4.19}$$

Expressing $Z(\beta)$ by its modular transformation with the approximation for $Z(\tilde{\beta})$ gives:

$$\begin{aligned} d(N) = \frac{1}{2\pi i} \int d\beta\, e^{\beta N} \Big(\frac{\beta}{2\pi}\Big)^{12} Z(\tilde{\beta}) \;&\sim\; \frac{1}{2\pi i} \int d\beta \,\Big(\frac{\beta}{2\pi}\Big)^{12} e^{\beta N + \frac{4\pi^2}{\beta}} \\ &= \frac{1}{2\pi i}\, \frac{1}{(2\pi)^{12}} \int d\beta\, e^{\beta N + \frac{4\pi^2}{\beta} + 12\log\beta} \end{aligned} \tag{4.20}$$

We can approximate this integral by using the saddle point method. First, notice that the integral has the general form $\int d\beta\, e^{f(\beta)}$. The function $f(\beta)$ has a maximum where $f'(\beta) = 0$. Now approximate $f(\beta)$ with its Taylor series up to quadratic terms:

$$f(\beta) \approx f(\beta_0) + \frac{1}{2}(\beta - \beta_0)^2 f''(\beta_0) \tag{4.21}$$

$$f'(\beta) = N - \frac{4\pi^2}{\beta^2} + \frac{12}{\beta} = 0 \tag{4.22}$$

The second condition gives an estimate of $\beta_0$ for large $N$:

$$\beta_0 = \frac{2}{N}(\sqrt{N\pi^2 + 9} - 3) \approx \frac{2\pi}{\sqrt{N}} \tag{4.23}$$

The Taylor series for the exponent function then reads:

$$f(\beta) \approx 4\pi\sqrt{N} + 12\log\Big(\frac{2\pi}{\sqrt{N}}\Big) + \frac{1}{2}\Big(\beta - \frac{2\pi}{\sqrt{N}}\Big)^2 \Big(\frac{N\sqrt{N}}{\pi} + \frac{3N}{\pi^2}\Big) \tag{4.24}$$

The first two terms are independent of $\beta$ and can be pulled out of the integral, while the last term is approximated and all terms sub-leading in $N$ are discarded for very large N:

$$\frac{1}{2}\Big(\beta - \frac{2\pi}{\sqrt{N}}\Big)^2\Big(\frac{N\sqrt{N}}{\pi} + \frac{3N}{\pi^2}\Big) \approx \frac{N\sqrt{N}}{2\pi}\beta^2 \tag{4.25}$$

The expression for $d(N)$ now has a simplified form in terms of a Gaussian integral:

$$d(N) \approx \frac{1}{2\pi i}\, e^{4\pi\sqrt{N}}\frac{1}{N^6}\lim_{T\to\infty}\int_{\epsilon - iT}^{\epsilon + iT} d\beta\, \exp\Big(\frac{N\sqrt{N}}{2\pi}\beta^2\Big) \tag{4.26}$$

Using a complex rotation $\beta \to i\beta$ and letting $\epsilon \to 0$, this can be evaluted over $\mathbb{R}$ to yield:

$$d(N) \approx e^{4\pi\sqrt{N}}\frac{\sqrt{\pi}}{2\pi N^6}\sqrt{\frac{2\pi}{N\sqrt{N}}} = \frac{1}{\sqrt{2}}\, e^{4\pi\sqrt{N}} N^{-\frac{27}{4}} \tag{4.27}$$

This finally gives us an expression for the microscopic entropy:

$$S = \log d(N) \approx 4\pi\sqrt{N} - \frac{27}{2}\log(\sqrt{N}) + ... \tag{4.28}$$

Note that the first term is reproduced by the Cardy formula, which states that we can approximate the degeneracy of $c = 24$ left-moving bosons in the conformal field theory by

$$Z(q) = Tr(q^{L_0 - \frac{c}{24}}) \quad \to \quad d(N) \sim \exp\Big(2\pi\sqrt{\frac{cN}{6}}\Big) \tag{4.29}$$

Considering (4.1), we conclude that the entropy for a given charge vector is given by:

$$S(Q) := \log d(Q) \sim 4\pi\sqrt{\frac{Q^2}{2}} - \frac{27}{4}\log(\frac{Q^2}{2}) \tag{4.30}$$

The first term in this approximation corresponds exactly to the Wald entropy of a black hole with electric charges $Q$ and no magnetic charge, where higher-order curvature terms - conceptualized as "stringy" quantum corrections - are included in the gravity action.

## 4.2. Type II strings and the D1-D5-P system

We have discussed in the previous chapter that there is a duality between heterotic string theory compactified on a torus $T^5$ and Type II string theory compactified on $K3 \times S^1$. We now look at the Type IIB construction that was first considered by Strominger and Vafa [1], but first considering the simpler toroidal compactification on $T^5$ mentioned in [8].

The physical construction of interest is characterized by the presence of D1- and D5-branes, which is necessary in Type II string theory for a solitonic BPS state. There are $Q_1$ units of charge, corresponding to $Q_1$ windings of D1-branes around a circle. There is an ambiguity whether this is achieved by $Q_1$ D1-branes wrapping the circle once or by a single D1-brane with winding number $Q_1$ (even something inbetween should be possible, as a long as we have a bound state). Furthermore, we also have $Q_5$ units of charge corresponding to one or more D5-branes, and $n$ units of momentum around the circle.

A low-energy CFT interpretation of such a bound state is that we have conformal fields corresponding to zero modes of open strings which connect the D1- and D5-branes. This system has the same excitations as those of a single string winding around the circle $Q_1Q_5$ times. The relevant bosonic fields correspond to the position of the string in the four transverse compact dimensions of $T^5 = T^4 \times S^1$. Requiring supersymmetry, we must also have the same number of fermionic fields. In summary, we consider a single string wound $Q_1Q_5$ times around an $S^1$ that is only allowed to oscillate in four transverse directions. The level-matching condition for the oscillator numbers of left- and right-movers is

$$N_L - N_R = nw = nQ_1Q_5 \tag{4.31}$$

where either $N_L$ or $N_R$ must vanish by requiring BPS states. First, let us write down the partition function for the free bosonic oscillators, of which we have four:

$$Z_{bos}(\beta) = \frac{1}{\eta(\tau)^4} = q^{-\frac{4}{24}} \prod_{s=1}^{\infty} \frac{1}{(1-q^s)^4} \tag{4.32}$$

The partition function for a free fermion in the Ramond sector is given by [29]

$$Z_f(\beta) = \sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}} = \sqrt{2}\, q^{\frac{1}{24}} \prod_{s=1}^{\infty}(1+q^s) \tag{4.33}$$

where $\vartheta_2(\tau)$ is a Jacobi theta function, yielding the partition function of the whole system:

$$Z_{BPS}(\beta) = 4 \prod_{s=1}^{\infty} \frac{(1+q^s)^4}{(1-q^s)^4} = \left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^2 \frac{1}{\eta(\tau)^4} \tag{4.34}$$

We can make use of some identities for modular forms (see Appendix B) to rewrite $Z_f$:

$$\eta\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\,\eta(\tau)\,, \quad \vartheta_4\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\,\vartheta_2(\tau) \tag{4.35}$$

The partition function is then simplified using the modular property of $\eta(\tau) = \Delta(q)^{\frac{1}{24}}$:

$$Z_{BPS}(\beta) = \frac{\vartheta_2(\tau)^2}{\eta(\tau)^6} = \frac{(\sqrt{-i\tau})^6}{(\sqrt{-i\tau})^2} \frac{\vartheta_4(-1/\tau)^2}{\eta(-1/\tau)^6} = (-i\tau)^2 \frac{\vartheta_4(-1/\tau)^2}{\eta(-1/\tau)^6} \tag{4.36}$$

With our definitions of $q = e^{-\beta} = e^{2\pi i\tau}$ and $\tilde{q} = e^{-\tilde{\beta}}$ with $\tilde{\beta} = 4\pi^2/\beta$, the partition function now reads:

$$Z_{BPS}(\beta) = \Big(\frac{\beta}{2\pi}\Big)^2 \frac{\vartheta_4(\tilde{\beta})^2}{\eta(\tilde{\beta})^6} =: \Big(\frac{\beta}{2\pi}\Big)^2 Z_{BPS}(\tilde{\beta}) \tag{4.37}$$

Using the definitions of $\vartheta_4$ and $\eta$, we write down the product expansion of $Z_{BPS}(\tilde{\beta})$:

$$Z_{BPS}(\tilde{\beta}) = \tilde{q}^{-\frac{1}{4}} \prod_{s=0}^{\infty} \frac{(1-\tilde{q}^{s+\frac{1}{2}})^4}{(1-\tilde{q}^{s+1})^4} \tag{4.38}$$

Around $\tilde{q} = 0$, this is approximated by the series expansion

$$Z_{BPS}(\tilde{\beta}) \approx \tilde{q}^{-\frac{1}{4}} - 4\tilde{q}^{\frac{1}{4}} + 10\tilde{q}^{\frac{3}{4}} + \mathcal{O}(\tilde{q}^{\frac{5}{4}}) \tag{4.39}$$

which allows us to approximate the partition function in the low temperature limit $\tilde{q} \to 0$:

$$Z_{BPS}(\tilde{\beta}) \sim \tilde{q}^{-\frac{1}{4}} = e^{\pi^2/\beta} \tag{4.40}$$

We use this to evalute the degeneracy by an inverse Laplace transform as in section 4.1:

$$d(N) = \frac{1}{2\pi i} \int d\beta \, e^{\beta N} \left(\frac{\beta}{2\pi}\right)^2 Z_{BPS}(\tilde{\beta}) \tag{4.41}$$

$$\sim \frac{1}{2\pi i} \frac{1}{(2\pi)^2} \int d\beta \, e^{\beta N + \frac{\pi^2}{\beta} + 2\log\beta} \tag{4.42}$$

Using the saddle-point method for large $N$ as before, we approximate as follows:

$$f(\beta) \approx f(\beta_0) + \frac{1}{2}(\beta - \beta_0)^2 f''(\beta_0) \tag{4.43}$$

$$f'(\beta) = N - \frac{\pi^2}{\beta^2} + \frac{2}{\beta} = 0 \quad \to \quad \beta_0 \approx \frac{\pi}{\sqrt{N}} \tag{4.44}$$

The Taylor series for the exponent function then reads after approximating for large $N$:

$$f(\beta) \approx 2\pi\sqrt{N} + 2\log\left(\frac{\pi}{\sqrt{N}}\right) + \frac{N\sqrt{N}}{\pi}\beta^2 \tag{4.45}$$

This gives us the following expressions for the degeneracy and the entropy:

$$d(N) \approx e^{2\pi\sqrt{N}} N^{-7/4} \tag{4.46}$$

$$S = \log d(N) \approx 2\pi\sqrt{N} - \frac{7}{4}\log(N) + ... \tag{4.47}$$

The entropy again is proportional to $\sqrt{N}$, which is dependent on the charges because we have $N = nQ_1Q_5$. This is equivalent to the entropy of the Strominger-Vafa black hole as computed in [1] via the Cardy formula, which we will discuss in the following section. The Cardy formula, as stated in the previous section, reads:

$$d(N) \sim \exp\left(2\pi\sqrt{\frac{cN}{6}}\right) \tag{4.48}$$

Since we have four free bosonic and fermionic fields each - which in the conformal field theory have central charges $c = 1$ and $c = 1/2$, respectively [29] - the central charge of this system is $c = 4 + 2 = 6$, giving the approximation

$$d(N) \sim \exp\left(2\pi\sqrt{N}\right) \tag{4.49}$$

which gives a relation for the entropy of the Strominger-Vafa black hole:

$$S \sim 2\pi\sqrt{N} \tag{4.50}$$

We also have a logarithmic second term, which corresponds to the first stringy correction to the macroscopic entropy of a black hole [8].

## 4.3. The Strominger-Vafa black hole in D=5

Consider the low-energy effective action of Type II string theory compactified on $K3 \times S^1$:

$$S = \frac{1}{16\pi} \int d^5x \sqrt{-\tilde{g}} \left( e^{-2\phi}(R + 4(\nabla\phi)^2 - \frac{1}{4}\tilde{H}^2) - \frac{1}{4}F^2 \right) \tag{4.51}$$

where $F$ is a RR 2-form field strength and $\tilde{H}$ is a 2-form axion field strength which comes from the NS-NS 3-form where one component is associated with the tangential circle $S^1$ from the compactification. In the Einstein frame $g = e^{-4\phi/3}\tilde{g}$ we can rewrite this as:

$$S = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left( R + \frac{4}{3}(\nabla\phi)^2 - \frac{e^{-4\phi/3}}{4}\tilde{H}^2 - \frac{e^{2\phi/3}}{4}F^2 \right) \tag{4.52}$$

The fields have electric charges, which are evaluated by integrals over a surrounding $S^3$:

$$Q_H = \frac{1}{4\pi^2} \int_{S^3} \star e^{-4\phi/3}\tilde{H} \tag{4.53}$$

$$Q_F = \frac{1}{16\pi} \int_{S^3} \star e^{2\phi/3} F \tag{4.54}$$

The equations of motion have a spherically symmetric solution which is the Reissner-Nordström black hole in $D = 5$ with a charge $Q \sim (Q_H Q_F^2)^{1/3}$ with the horizon:

$$r_0 = \left( \frac{8 Q_H Q_F^2}{\pi^2} \right)^{1/6} \tag{4.55}$$

The Bekenstein-Hawking entropy of this black hole is then

$$S_{BH} = 2\pi \sqrt{\frac{Q_H Q_F^2}{2}} \tag{4.56}$$

The string theory side of the D1-D5-P construction on $K3 \times S^1$ can be understood as a sigma model with a target space of $k = \frac{1}{2}Q_F^2 + 1$ copies of $K3$. We count quarter-BPS states on M, where states in the right-moving RR sector are again kept in the ground state. This is described by a conformal field theory with central charge $c = 6k$. The momentum $P$ on the $S^1$ is then equal to the left-moving oscillator number $N$ and further related to the charge of $\tilde{H}$ by $P = Q_H$. Application of the Cardy formula then yields:

$$S = \log d(Q_F, Q_H) \sim 2\pi \sqrt{Q_H \left( \frac{1}{2} Q_F^2 + 1 \right)} \tag{4.57}$$

For large $Q_F$, this obviously reduces to the Bekenstein-Hawking entropy (4.56).

## 4.4. Dyonic quarter-BPS black holes in D=4

We now briefly sketch another construction, which is related to the D1-D5-P system in five dimensions. We are considering quarter-BPS states in $D = 4$ Type IIB string theory compactified on $K3 \times S^1 \times \tilde{S}^1$ with the following charges, as reviewed in [4]:

- 1 Kaluza-Klein monopole associated with the circle $\tilde{S}^1$
- 1 D5-brane wrapping $K3 \times S^1$
- $m$ D1-branes wrapping $S^1$
- $n$ units of momentum along $S^1$
- $l$ units of momentum along $\tilde{S}^1$

Using the fact that the Kaluza-Klein geometry has the asymptotic limit $\mathbb{R}^3 \times \tilde{S}^1$ at $r \to \infty$ and $\mathbb{R}^4$ at the monopole $r \to 0$, we can relate this construction to the $D = 5$ black hole construction investigated by Strominger and Vafa. This connection is called a 4D-5D lift. Note that in this case we have electric as well as magnetic charges, which is why we call this a *dyonic* state. A dyonic state is characterized by a charge vector which takes the form $\Gamma^i_\alpha = (Q^i, P^i)$ where $Q^i$ and $P^i$ are the electric and magnetic charges. The index $i = 1, ..., 28$ transforms under the previously mentioned T-duality group $O(22, 6; \mathbb{Z})$ while $\alpha = 1, 2$ transforms under the S-duality group $SL(2, \mathbb{Z})$. There are three quantities formed by the charges which are invariant under T-duality transformations:

$$(P^2/2,\ Q^2/2,\ P \cdot Q) := (m, n, l) \tag{4.58}$$

We introduce the chemical potentials $(\sigma, \tau, z)$ and their corresponding fugacites $(p, q, y)$:

$$Z(\sigma, \tau, z) = \sum_{m,n,l} p^m q^n y^l (-1)^l \, d(m, n, l) \tag{4.59}$$

For large radius $R$ and weak coupling, this can be decomposed as

$$Z(\Omega) = Z_{D1}(p, q, y) \, Z_{KK}(q) \, Z_{CM}(q, y) \tag{4.60}$$

where $Z_{D1}(p, q, y)$ counts bound states of D1-branes bound to a single D5-brane, $Z_{KK}(q)$ counts bound states of the Kaluza-Klein monopole and $Z_{CM}(q, y)$ counts bound states of center-of-mass motion in the Taub-NUT space. We get the following for these factors:

$$Z_{D1}(p, q, y) = \frac{1}{p} \prod_{s,t,r} \frac{1}{(1 - p^s q^t y^r)^{c(s,t,r)}} \tag{4.61}$$

$$Z_{KK}(q) = \frac{1}{q} \prod_{s=1}^{\infty} \frac{1}{(1 - q^s)^{24}} = \frac{1}{\eta(q)^{24}} \tag{4.62}$$

$$Z_{CM}(q, y) = \frac{\eta^6(q)}{\theta_1^2(p, y)} \tag{4.63}$$

where again $\eta(q)$ is the Dedekind $\eta$ function and $\theta_1^2(p, y)$ is a Jacobi theta form (see Appendix B). The combined partition function also has an expression in terms of a modular form, namely the Igusa cusp form $\Phi_{10}(\Omega)$, which is the inverse of the partition function:

$$Z(\Omega) = \frac{1}{\Phi_{10}(\Omega)} \tag{4.64}$$

# A. Calabi-Yau manifolds

A complex manifold is an even-dimensional topological space $M$ with an atlas $(U_i, \phi_i)$ where $U_i$ is an open cover of $M$ such that $M = \bigcup_i U_i$ and the maps $\phi_i$ are homeomorphisms from $U_i$ to an open subset of $\mathbb{C}^n$. For any open sets $U_i, U_j$ on $M$ the transition functions $\psi_{ji} = \phi_j \circ \phi_i^{-1}$ are holomorphic. It is interesting to note that a holomorphic function $f : M \to \mathbb{C}$ on a compact complex manifold is constant. For a complex manifold with dimension $m = dim_{\mathbb{C}} M$, the tangent space $T_pM$ is spanned by a $2m$-dimensional basis

$$\left\{ \frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^m}; \frac{\partial}{\partial \bar{z}^1}, ..., \frac{\partial}{\partial \bar{z}^m} \right\} \tag{A.1}$$

while the cotangent space $T_p^* M$ is spanned by

$$\{dz^1, ..., dz^m; \, d\bar{z}^1, ..., d\bar{z}^m\} \tag{A.2}$$

where the $\frac{\partial}{\partial z^k}, dz^k$ are holomorphic and the $\frac{\partial}{\partial \bar{z}^k}, d\bar{z}^k$ are anti-holomorphic basis elements.

An almost complex structure is a linear map $J_p : T_pM \to T_pM$ such that $J_p^2 = -id_{T_pM}$. In terms of the basis elements, it is given by:

$$J_p = i \, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i \, d\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu} \tag{A.3}$$

A Hermitian metric $g$ is a Riemannian metric on a complex manifold $M$ that satisfies $g_p(J_p X, \, J_p Y) = g_p(X, Y)$ for every $p \in M$ and for any $X, Y \in T_pM$. Due to symmetry, the metric takes the form:

$$g = g_{\mu\bar{\nu}} \, dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\nu} \, d\bar{z}^\mu \otimes dz^\nu \tag{A.4}$$

A Kähler form $\Omega$ is the real tensor field $\Omega = i g_{\mu\bar{\nu}} \, dz^\mu \wedge d\bar{z}^\nu$. We can define a $2m$-form $\Omega \wedge ... \wedge \Omega$ that vanishes nowhere, indicating that every complex manifold is orientable.

A Kähler manifold is a Hermitian manifold whose Kähler form is closed, i.e. $d\Omega = 0$. Locally, on a given chart $U_i$, the Kähler metric $g$ can be expressed as $g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \mathcal{K}$ where $\mathcal{K}$ is called the Kähler potential. Furthermore, we locally have $\Omega = i\partial\bar{\partial} \, ln \, \mathcal{K}$ with the Dolbeault operators $\partial, \bar{\partial}$ taking $(r, s)$-forms to $(r+1, s)$- and $(r, s+1)$-forms, respectively.

The Ricci form $\mathfrak{R}$ is defined as $\mathfrak{R} = -i \, \partial_{\bar{\nu}} \partial_\mu \, ln \sqrt{g} \, dz^\mu \wedge d\bar{z}^\nu$. A Kähler metric with $\mathfrak{R} = 0$ is then called Ricci flat. If a Kähler manifold $M$ admits a Ricci flat metric, its first Chern class $c_1(M) = \mathfrak{R}/2\pi$ vanishes. A compact Kähler manifold with vanishing first Chern class and thus vanishing Ricci form is called a Calabi-Yau manifold. The restricted holonomy group of a $m$-dimensional Calabi-Yau manifold $M$ is contained in $SU(m) \subset SO(2m)$.

The simplest examples of Calabi-Yau manifolds are the torus in $m = 1$ and the K3 surfaces in $m = 2$, while for $m = 3$ there exists a very large number of Calabi-Yau threefolds [30].

# B. Modular forms

We summarize here a few basic aspects of modular forms as detailed in [4, 30, 31]:

The modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ generates the modular transformations:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \text{ with } (ad - bc) = 1 \tag{B.1}$$

A modular form $f(\tau)$ of weight $k$ on $SL(2,\mathbb{Z})$ is a holomorphic function on the upper half of the complex plane, denoted by $\mathbb{H}$ and satisfying $\text{Im}(\tau) > 0$, that transforms as:

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau) \tag{B.2}$$

It follows from these transformations that modular forms are periodic under $\tau \to \tau + 1$ and can be written as a Fourier series:

$$f(\tau) = \sum_{n=-\infty}^{\infty} a(n)\, q^n \ , \quad q := e^{2\pi i \tau} \tag{B.3}$$

Modular forms that vanish at infinity and for which $a(0) = 0$ are called cusp forms. An important modular form on $SL(2,\mathbb{Z})$ is the discriminant function $\Delta(\tau)$, which is a modular form of weight $k = 12$, related to the Dedekind eta function $\eta(\tau)$:

$$\Delta(\tau) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \eta(\tau)^{24} \tag{B.4}$$

A Jacobi form is a holomorphic function $\phi(\tau, z) : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ that is modular in $\tau$ and elliptic in $z$, meaning it transforms under modular transforms as

$$\varphi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{B.5}$$

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2 \tau + 2\lambda z)} \varphi(\tau, z) \qquad \forall \ \lambda, \mu \in \mathbb{Z} \tag{B.6}$$

which implies that it is periodic under $\tau \to \tau + 1, z \to z + 1$ and has a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n, r)\, q^n y^r \ , \quad q := e^{2\pi i \tau},\ y := e^{2\pi i z} \tag{B.7}$$

where we have $c(n, r) = 0$ unless $4mn \geq r^2$ holds. The Jacobi theta function is defined as

$$\theta[{}^a_b](v, \tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-a)^2} e^{2\pi i (v-b)(n-a)} \tag{B.8}$$

Furthermore, we define the functions $\vartheta_1(\tau), \vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau)$ as:

$$\vartheta_1(\tau) = \theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](\tau,0) = 0 \tag{B.9}$$

$$\vartheta_2(\tau) = \theta\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right](\tau,0) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} = 2\eta(\tau)\, q^{\frac{1}{12}} \prod_{r=1}^{\infty}(1+q^r)^2 \tag{B.10}$$

$$\vartheta_3(\tau) = \theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\tau,0) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} = \eta(\tau)\, q^{-\frac{1}{24}} \prod_{r=0}^{\infty}(1+q^{r+\frac{1}{2}})^2 \tag{B.11}$$

$$\vartheta_4(\tau) = \theta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right](\tau,0) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} = \eta(\tau)\, q^{-\frac{1}{24}} \prod_{r=0}^{\infty}(1-q^{r+\frac{1}{2}})^2 \tag{B.12}$$

The functions satisfy the following relations under modular transformations:

$$\vartheta_2\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\,\vartheta_4(\tau)\,, \qquad \vartheta_4\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\,\vartheta_2(\tau) \tag{B.13}$$

The modular transformation of the Dedekind eta function is given by:

$$\eta\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\,\eta(\tau) \tag{B.14}$$

An important relation that follows from the product representations of these functions is:

$$\sqrt{\frac{\vartheta_3\vartheta_4\vartheta_2}{2\eta^3}} = \prod_{r=1}^{\infty}\frac{(1-q^r)}{(1-q^{2r})}(1+q^r) = 1 \tag{B.15}$$

This allows us to express $\vartheta_2(\tau)$ in terms of the Dedekind eta function $\eta(\tau)$:

$$\sqrt{\vartheta_2(\tau)} = \sqrt{2}\,\eta(\tau)\, q^{\frac{1}{24}} \prod_{r=1}^{\infty}(1+q^r) = \frac{\sqrt{2}\,\eta(2\tau)}{\eta(\tau)} \tag{B.16}$$

Let M be an element of the symplectic group $Sp(2,\mathbb{Z})$, written in the form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \forall\, (A,B,C,D) \in SL(2,\mathbb{Z}) \tag{B.17}$$

Let $\mathbb{H}_2$ be the Siegel upper half plane, defined as the set of $2\times 2$ matrices $\Omega$

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \qquad \forall\, \tau, z, \sigma \in \mathbb{C} \tag{B.18}$$

where we have the constraints $\text{Im}(\tau) > 0,\ \text{Im}(\sigma) > 0,\ \det \text{Im}(\Omega) > 0.$

A Siegel form $F(\Omega) : \mathbb{H}_2 \to \mathbb{C}$ of weight $k$ is then a holomorphic function satisfying:

$$F[(A\Omega + B)(C\Omega + D)^{-1}] = [det(C\Omega + D)]^k F(\Omega) \tag{B.19}$$

This again has an expression in terms of a Fourier series:

$$F(\Omega) = \sum_{n,r,m} a(n,r,m)\, q^n y^r p^m\,, \quad q := e^{2\pi i\tau},\ y := e^{2\pi i z},\ p := e^{2\pi i\sigma} \tag{B.20}$$

An example of a Siegel modular form is the Igusa cusp form $\Phi_{10}$ with weight 10, which appears in the partition function of dyons in $\mathcal{N} = 4$ string theory.

# Bibliography

[1] Andrew Strominger and Cumrun Vafa. *Microscopic origin of the Bekenstein-Hawking entropy. Phys. Lett.*, B379:99–104, 1996, `arXiv:hep-th/9601029` [hep-th].

[2] Ipsita Mandal and Ashoke Sen. *Black Hole Microstate Counting and its Macroscopic Counterpart. Nucl. Phys. Proc. Suppl.*, 216:147–168, 2011, `arXiv:1008.3801` [hep-th]. [Class. Quant. Grav.27,214003(2010)].

[3] Ashoke Sen. *Microscopic and Macroscopic Entropy of Extremal Black Holes in String Theory. Gen. Rel. Grav.*, 46:1711, 2014, `arXiv:1402.0109` [hep-th].

[4] Atish Dabholkar and Suresh Nampuri. *Quantum black holes. Lect. Notes Phys.*, 851:165–232, 2012, `arXiv:1208.4814` [hep-th].

[5] Robert C. Myers and M. J. Perry. *Black Holes in Higher Dimensional Space-Times. Annals Phys.*, 172:304, 1986.

[6] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli. *The Black hole in three-dimensional space-time. Phys. Rev. Lett.*, 69:1849–1851, 1992, `arXiv:hep-th/9204099` [hep-th].

[7] J. David Brown, M. Henneaux, and C. Teitelboim. *Black Holes in Two Space-time Dimensions. Phys. Rev.*, D33:319–323, 1986.

[8] K. Becker, M. Becker, and J. H. Schwarz. *String theory and M-theory: A modern introduction.* Cambridge University Press, 2006.

[9] Alexander M. Polyakov. *Quantum Geometry of Bosonic Strings. Phys. Lett.*, 103B:207–210, 1981.

[10] Augusto Sagnotti. Open Strings and their Symmetry Groups. In *NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory (Cargese Summer Institute) Cargese, France, July 16-30, 1987*, pages 0521–528, 1987, `arXiv:hep-th/0208020` [hep-th].

[11] Atish Dabholkar. Lectures on orientifolds and duality. In *High-energy physics and cosmology. Proceedings, Summer School, Trieste, Italy, June 2-July 4, 1997*, pages 128–191, 1997, `arXiv:hep-th/9804208` [hep-th].

[12] Edward Witten. *String theory dynamics in various dimensions. Nucl. Phys.*, B443:85–126, 1995, `arXiv:hep-th/9503124` [hep-th].

[13] E. Cremmer, B. Julia, and Joel Scherk. *Supergravity Theory in Eleven-Dimensions. Phys. Lett.*, 76B:409–412, 1978.

[14] W. Nahm. *Supersymmetries and their Representations. Nucl. Phys.*, B135:149, 1978.

[15] John H. Schwarz. *The power of M theory.* *Phys. Lett.*, B367:97–103, 1996, `arXiv:hep-th/9510086` [hep-th].

[16] Michael J. Duff. *The Theory formerly known as strings.* *Sci. Am.*, 278:64–69, 1998. [Spektrum Wiss.1998N4,62(1998)].

[17] J. Baez and J. P. Muniain. *Gauge fields, knots and gravity.* 1995.

[18] A. Font and S. Theisen. *Introduction to string compactification.* *Lect. Notes Phys.*, 668:101–181, 2005.

[19] Theodor Kaluza. *Zum Unitätsproblem der Physik.* *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1921:966–972, 1921.

[20] D. Wuensch. *The fifth dimension: Theodor Kaluza's ground-breaking idea.* *Annalen Phys.*, 12:519–542, 2003.

[21] Fernando Quevedo, Sven Krippendorf, and Oliver Schlotterer. *Cambridge Lectures on Supersymmetry and Extra Dimensions.* 2010, `arXiv:1011.1491` [hep-th].

[22] Steven Weinberg. *The quantum theory of fields. Vol. 3: Supersymmetry.* Cambridge University Press, 2013.

[23] E. B. Bogomolny. *Stability of Classical Solutions.* *Sov. J. Nucl. Phys.*, 24:449, 1976. [Yad. Fiz.24,861(1976)].

[24] M. K. Prasad and Charles M. Sommerfield. *An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon.* *Phys. Rev. Lett.*, 35:760–762, 1975.

[25] Joseph Polchinski. *Dirichlet Branes and Ramond-Ramond charges.* *Phys. Rev. Lett.*, 75:4724–4727, 1995, `arXiv:hep-th/9510017` [hep-th].

[26] Joseph Polchinski, Shyamoli Chaudhuri, and Clifford V. Johnson. *Notes on D-branes.* 1996, `arXiv:hep-th/9602052` [hep-th].

[27] David J. Gross and Malcolm J. Perry. *Magnetic Monopoles in Kaluza-Klein Theories.* *Nucl. Phys.*, B226:29–48, 1983.

[28] John H. Schwarz. *Lectures on superstring and M theory dualities: Given at ICTP Spring School and at TASI Summer School.* *Nucl. Phys. Proc. Suppl.*, 55B:1–32, 1997, `arXiv:hep-th/9607201` [hep-th].

[29] Ralph Blumenhagen and Erik Plauschinn. *Introduction to conformal field theory.* *Lect. Notes Phys.*, 779:1–256, 2009.

[30] M. Nakahara. *Geometry, Topology and Physics, Second Edition.* Graduate student series in physics. Taylor & Francis, 2003.

[31] H. Weber. *Lehrbuch der Algebra: Band 3. Elliptische Funktionen und algebraische Zahlen.* Lehrbuch der Algebra. F. Vieweg und Sohn, 1908.