NONLINEAR SYSTEM MODAL IDENTIFICATION USING CONTINUOUS WAVELET TRANSFORM

Mansour Nikkhah Mechanical Engineering Dept., University of Tehran, Iran

Arman Hajati Mechanical Engineering Dept., University of Tehran, Iran

mbahrami@ut.ac.ir

Abstract

In this paper *continuous wavelet transform* (CWT) is used to identify nonlinear stiffness in mechanical structures. The time-frequency representation of the free decay response of the system that results from wavelet transform presents information about localized periodicity resulted from exciting the nonlinearities. The nonlinear stiffness in the structure can be modeled by an equivalent time-varying linear stiffness which is a function of response amplitude. Accordingly, the *equivalent eigenfrequencies* of the structure are also obtained as a function of response amplitude. This phenomenon can be easily seen in the time-frequency domain maps of the system's response. As a result, the nonlinearity can be determined by studying the changes in the frequency spectrum versus time. This approach has been used to identify *MDOF* nonlinear system such as weakly damped structures with cubic stiffness nonlinearity.

1 Introduction

The ability to identify a mechanical structure is of great interest in vibration, which is one of the main goals of experimental studies on structural vibration. As an example in the case of linear systems, the modal testing is a well-known method for this purpose, in which the modal parameters such as natural frequencies, damping factors, and modal scaling of the system are determined by way of an experimental approach. For the identification of nonlinear structures, various procedures have been developed in mechanical vibration; a summary of them can be found in ref [1]. One of the most common approaches is to estimate a model based on conventional system identification algorithms; however, these methods face problem estimating low damped structures and lag terms.

The idea of applying wavelet analysis in vibration was first proposed by Newland (ref [2]). By applying CWT, some features can be extracted from time-frequency maps of transient response. Such information has been used to develop system identification algorithms in the time-frequency domain (ref [3]). The eigenfrequencies and damping coefficients of a TV tower excited by random wind effects were obtained using the wavelet transform by Gouttebroze and Lardies (ref [4]). Le and Argoul introduced a complete procedure for modal identification of a linear mechanical system using wavelet analysis of its free decay response (ref [5]).

The procedure of nonlinear system identification based on the ridges and skeletons of the wavelet transform was validated using a *SDOF* nonlinear experimental analysis in ref [6]. Lind et al. introduced wavelet analysis for processing flight data to extract information about structural nonlinearities and to predict limit cycles (ref [7]). However in these references, an approach for identification of nonlinearities in MDOF systems has not seriously tackled. Moreover, time-frequency analysis has been used in the context of nonlinear system identification as a tool to detect or to classify the nonlinearity (ref [8]); therefore, the aim of this paper is to improve the application of continuous wavelet transform to quantitatively identify nonlinear stiffness in MDOF mechanical structures.

2 Continuous Wavelet Transform

This section introduces briefly the continuous wavelet transform; more information about the transform can be found in ref [9]. Wavelets provide a means of analyzing time and frequency content in data by transforming a non-stationary signal from its input time domain to a time-scale domain. This type of processing decomposes the data into localized waveforms rather than the sum of infinite-length sinusoids that results from Fourier processing. Wavelet analysis relies upon a family of basis functions, called wavelets, for signal processing in the time-frequency domain. Wavelets can be essentially considered as finite-duration waveforms for the limited purposes of this paper. Through translation and dilatation operations, the wavelet transform decomposes the signal according to a set of functions deduced from a defined prototype function, assumed to be well localized in both time and frequency domains. Each element, $\psi_{a,\tau}(t)$, of the family of wavelets used for analysis is written with respect to a mother wavelet $\psi(t)$. Essentially, the positive scalar a, called scale or dilatational parameter, is used to stretch the wavelet and consequently increase the period and decrease the frequency components. In addition, the scalar τ, called translational parameter, is used to shift the wavelet in time. Thus, the wavelet elements can be explicitly written in terms of the mother wavelet:

$$\psi_{a,\tau}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-\tau}{a}\right) \tag{1}$$

The wavelet transform results in a three-dimensional map, in which $W(a,\tau)$ relates a measure of how much the signal, $x(t)$, correlates with $\psi_{a,\tau}(t)$ for each value of scale and position.

$$W(a,\tau) = \int_{-\infty}^{+\infty} x(t)\psi^{*}_{a,\tau}dt. \tag{2}$$

There is a multitude of wavelets that could be used; however, the Morlet wavelet is particularly attractive for analyzing signals that are generated by dynamical system. First, damped sinusoids are common responses from certain dynamical systems, so it is natural to correlate these responses with Morlet wavelets. Second, the Morlet wavelet has a single frequency. If a signal correlates highly with a scaled Morlet wavelet, then the frequency of the wavelet indicates the frequency of some dynamic that generated the signal. For that reason, the Morlet wavelet, defined as exponentially decaying sinusoidal signal, will be used throughout this paper.

$$\psi(t) = e^{\frac{-t^2}{2}}\cos 5t \tag{3}$$

2.1 Dominant Scale

Wavelet analysis may seem more difficult to use than traditional Fourier analysis because it is difficult to interpret the wavelet parameters in terms of commonly understood concepts. For example, it is straightforward to identify a natural frequency from a spectral plot, but this information is more difficult to identify on a wavelet maps. However, some features can be extracted from a wavelet map such as the dominant scale in the signal at each point in time. Consider a vertical strip, $W(a,\overline{\tau})$, taken from a three-dimensional map. This strip represents the correlations between the signal and wavelets that are centered at position $\overline{\tau}$ for all scale values given by the vector A. Define $\overline{W}$ as the peak magnitude of this correlation that occurs for the wavelet with scale of $\overline{a}$:

$$\overline{W} = W(\overline{a},\overline{\tau}) = \max_{a\in A} W(a,\overline{\tau}) = \max_{a\in A} W(a,\tau)\Big|_{\tau=\overline{\tau}} \tag{4}$$

The dominant scale is not identical to the concept of frequency; however, they can be closely related in the Morlet wavelet. Consider a given signal of frequency ω_{sig} which is sampled with the sampling time of T_{sig} . The wavelet map that is generated by analyzing the signal will show a peak at the scale value of $\overline{a}$. This scale is determined by the frequency, ω_{wave}, and the sampling period, T_{wave}, of the mother wavelet. Note that $T_{sig} = 0.01s$, $T_{wave} = 1s$ and $\omega_{wave} = 5\dfrac{rad}{s}$, in this paper:

$$\overline{a} = \frac{T_{wave}}{T_{sig}}\frac{\omega_{wave}}{\omega_{sig}} = \frac{500}{\omega_{sig}\left(\dfrac{rad}{s}\right)} \tag{5}$$

Now, consider the particular case of the free response of a viscously damped single degree of freedom linear system:

$$x(t) = Xe^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t+\theta_0\right) \tag{6}$$

The wavelet transform of $x(t)$ can be approximated by means of asymptotic techniques and the dominant scale and the corresponding wavelet modulus can be calculated as:

$$\overline{a} = \frac{500}{\omega_n\sqrt{1-\zeta^2}}, \left|\overline{W}\right| = X\sqrt{\overline{a}}\,e^{-\zeta w_n b} \tag{7}$$

Thus, the natural frequency of the system can be found by means of the dominant scale concept; and the corresponding damping ratio can be evaluated from the slope of the straight line of the logarithm of the wavelet modulus for the given value of the dominant scale. The same procedure can be applied for MDOF linear system by estimating the dominant scales and their corresponding damping ratios.

In order to show the procedure, consider a 2 degrees of freedom viscously damped linear system in which $\omega_1 = 25$, $\omega_2 = 15$, $\zeta_1 = 0.07$, and $\zeta_2 = 0.05$. The free decay response and its corresponding frequency response and wavelet transform are shown in figure 1.

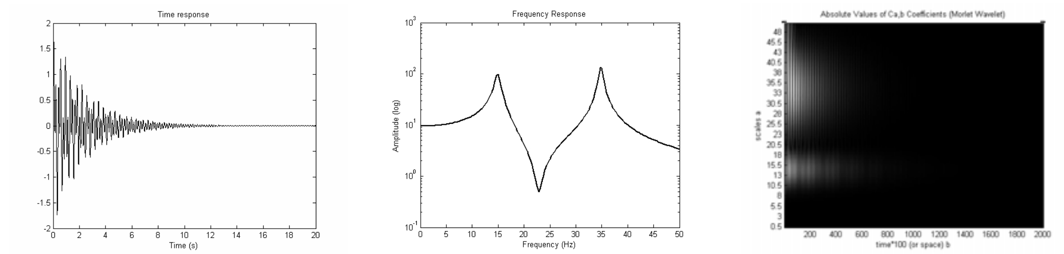

Figure 1. Time response of linear system and its corresponding Fourier and Wavelet transform

2.2 Modified Morlet Wavelet

For a better resolution of closely spaced modes, Gouttebroze and Lardies proposed a modified Morlet wavelet function defined by:

$$\psi_m(t) = e^{\frac{-t^2}{N}} \cos 5t \tag{8}$$

In the case of $N = 2$, we obtain the ordinary Morlet wavelet; but if we increase N, a narrower spectrum is obtained which allows a better resolution of closely spaced modes, but at the expense of lower time resolution. Indeed, increasing N will increase the frequency resolution but it decreases the time resolution. Compared to the conventional Morlet wavelet function, the modified one offers a better compromise in terms of localization for a signal, both in time domain and frequency domain. Consequently, the process is repeated using the modified Morlet wavelet $(N = 20)$ to achieve a better time-frequency map of the system's response as shown in figure 2.

Figure 2. Modified Morlet Wavelet $(N = 20)$ in comparison with the original one

3 Nonlinear System Identification

As it was declared, the dominant scales can be used to identify the dominant frequencies at distinct times during the responses. These frequencies may correspond to natural frequencies of dynamics that dominate the response. This time-varying nature of the dominant scale can indicate that the properties of the dominant dynamics are changing during the response. In this way, a simple evaluation of the time-frequency wavelet map can reveal information about the system dynamics that goes beyond the information obtained from a spectral plot.

Several features and trends are seen in wavelet maps from transient responses that indicate the presence of nonlinearities. Furthermore, these features and trends are exploited to describe the nature of the nonlinearity. Such information can potentially be used to develop nonlinear system identification algorithms in the time-frequency domain.

3.1 Equivalent Linear Stiffness and Eigenfrequencies for Nonlinear Stiffness

Now, consider the free decay response of a simple damped nonlinear system with cubic stiffness nonlinearity, in which $m = 1kg$, $c = 0.1\frac{N.s}{m}$, $k_l = 25\frac{N}{m}$, $k_n = 4\frac{N}{m^3}$; the time response and its corresponding frequency response and wavelet transform are shown in figure 3.

$$m\ddot{x} + c\dot{x} + k_l x + k_n x^3 = 0 \tag{9}$$

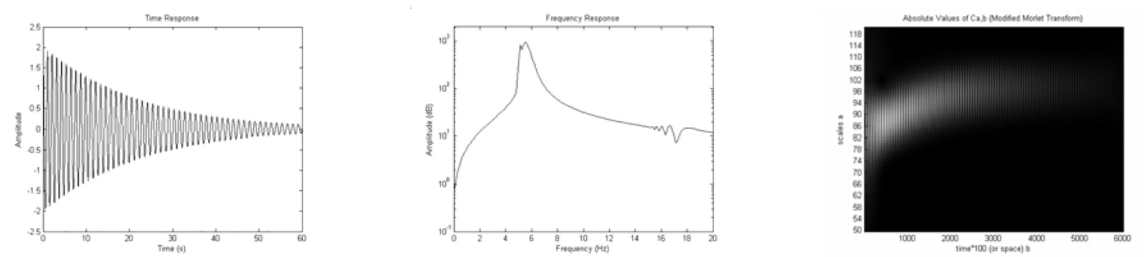

Figure 3. Time response of SDOF nonlinear system and its corresponding Fourier and Wavelet transform

Considering the frequency response, the system may be wrongly considered as a damped linear system whose natural frequency is about $5.5\frac{rad}{s}$. Conversely, the time-frequency representation of the system shows a remarkable time-varying dominant scale which can be exploited to identify the nonlinearity in the system. Consequently, while the spectral plot can be used as a powerful tool for linear system modal identification, it loses its performance in the case of nonlinear systems. On the other hand, the wavelet map reveals information about the time-varying system dynamics that can be used for the nonlinear system identification. At this moment, we investigate the time response and its wavelet maps to illustrate the concept. If we express the total force of springs as (10), an equivalent linear stiffness that is a function of amplitude can be assumed for the system.

$$F_{spring} = k_l x + k_n x^3 = \left(k_l + k_n x^2\right) x = k_{eq}(x).x \tag{10}$$

A hardening spring shows a larger stiffness for large amplitude in comparison with the small oscillation in which the spring stiffness can be approximated by the linear part. Therefore, the system's eigenfrequency will be increased for large amplitude and this phenomenon can be easily seen in the time-frequency representation. The dominant scale is gradually increased from 85 up to 100 by decreasing the oscillation's amplitude; using eq 5, it means that the eigenfrequency of the system is decreased from 5.88 to 5. Considering the solution of *Duffing's equation* using perturbation method, we can express the equivalent linear stiffness for cubic nonlinearity for sinusoid motion:

$$x(t) = X\sin(\omega t)\,,\; k_{eq} = k_l + \frac{3}{4}k_n X^2 \qquad (11), (12)$$

For example, the amplitude of the previous system's response at $t = 10$ is about 1.3; therefore, the equivalent frequency of the system and its corresponding dominant scale can be obtained as:

$$\omega_{eq} = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{25 + \frac{3}{4} \cdot 2 \cdot 1.3^2}{1}} = 5.49 \; , \; \overline{a}_{eq} = \frac{500}{\omega_{eq}} = 91.2 \qquad (13), (14)$$

for which the result is identical to that of wavelet maps. Now consider the case of 2 degrees of freedom nonlinear damped system. The governing dynamic equations are:

$$\begin{cases} m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2\left(\dot{x}_1 - \dot{x}_2\right) + k_{l1}x_1 + k_{l2}\left(x_1 - x_2\right) + k_{n1}x_1^3 + k_{n2}\left(x_1 - x_2\right)^3 = 0 \\ m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) + k_{l2}\left(x_2 - x_1\right) + k_{n2}\left(x_2 - x_1\right)^3 = 0 \end{cases} \qquad (15)$$

In case of $m_1 = 2kg$, $m_2 = 1kg$, $c_1 = 0.15\frac{N.s}{m}$, $c_2 = 0.10\frac{N.s}{m}$, $k_{l1} = 10\frac{N}{m}$, $k_{l1} = 25\frac{N}{m}$, $k_{n1} = 4\frac{N}{m^3}$, $k_{n2} = 3\frac{N}{m^3}$ free decay responses of the system are evaluated numerically, considering the initial condition of $x_1(0) = 2$, $x_2(0) = -1$, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$; moreover, the wavelet maps of the responses are found as shown in figure 4.

Figure 4. Corresponding wavelet maps

The time-varying dynamics, resulted from system's nonlinearity, can be easily determined by investigating the time-representation of the response. The challenge would be to define the equivalent linearized stiffness of the system for the reason that the form of oscillation is no more sinusoid. Therefore, the equivalent sinusoid amplitude is proposed for the non-sinusoid response of the system as follow:

$$X_{eq}^{n} = \frac{\sqrt[n]{\dfrac{\int_T |x|^n(t)dt}{T}}}{\sqrt[n]{\dfrac{\int_T \left|\sin\left(\omega t\right)\right|^n dt}{T}}}, \qquad (16)$$

where n is the effective power factor. For example for the case of cubic nonlinearity, because of the cubic nature of the stiffness, $n = 3$ is proposed; to verify the correctness of this factor consider the time-frequency representation for the undamped response of the previous nonlinear system, shown in figure 5.

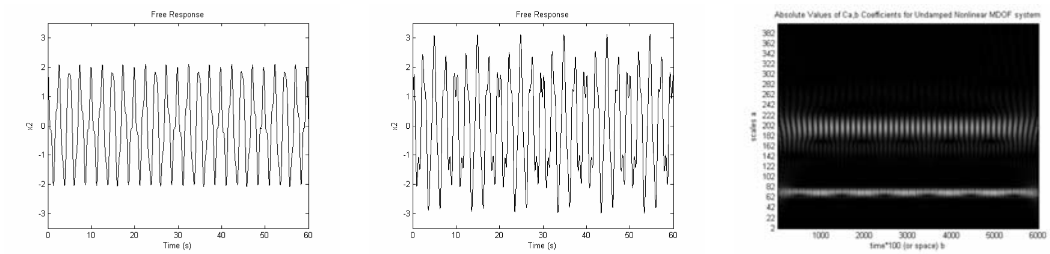

Figure 5. Free Responses of Undamped Nonlinear System and Corresponding Wavelet Map

Now we calculate the eigenfrequencies for this system using the idea of equivalent linearized stiffness to illustrate the procedure:

$$X_{1eq}^{III}=\frac{\sqrt[3]{\frac{\int_T |x_1|^3(t)dt}{T}}}{\sqrt[3]{\frac{\int_T |\sin(\omega t)|^3 dt}{T}}}=1.8367,X_{1-2eq}^{III}=\frac{\sqrt[3]{\frac{\int_T |x_1-x_2|^3(t)dt}{T}}}{\sqrt[3]{\frac{\int_T |\sin(\omega t)|^3 dt}{T}}}=1.2651 \tag{17}$$

$$k_{1eq}=k_{l1}+\frac{3}{4}k_{n1}X_{1eq}^2=20.1204,k_{2eq}=k_{l2}+\frac{3}{4}k_{n2}X_{1-2eq}^2=28.6011 \tag{18}$$

$$\omega_{1,2}^2=\sigma\left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}^{-1}\begin{bmatrix} k_{1eq}+k_{2eq} & -k_{2eq} \\ -k_{2eq} & k_{n2eq} \end{bmatrix}\right)=\begin{cases} 6.146 \\ 46.816 \end{cases} \tag{19}$$

The dominant scales in the wavelet maps are approximately 199 and 72 for the first and the second frequency, respectively. Consequently, the eigenfrequencies of the system are obtained:

$$\omega_{1,2}^2=\left(\frac{500}{199}\right)^2,\left(\frac{500}{72}\right)^2=6.313,48.225 \tag{20}$$

Therefore, the approximated eigenfrequencies of the system by means of equivalent linearized stiffness coincide with the real ones read from wavelet maps and the procedure is verified in this way.

3.2 Nonlinear Parameters Extraction

The aim of this section is to express the procedure of nonlinear stiffness parameters extraction from the wavelet maps. It is assumed that the time domain is divided into m windows, $T_1,T_2,\cdots,T_m$, and for each window the average equivalent amplitude, $X_{eq1,i}^{III},X_{eq1-2,i}^{III},X_{eq1,i}^{V},X_{eq1-2,i}^{V},X_{eq1,i}^{VII},X_{eq1-2,i}^{VII},\cdots$, is calculated using (16). Moreover, the average eigenfrequencies for each window is obtained from wavelet maps using (12), (13), (16) as $\omega_{1,2-1},\omega_{1,2-2},\cdots,\omega_{1,2-m}$. These average frequencies should be equal to the eigenvalues of the matrix $M^{-1}K$; therefore, these set of equations are obtained for each window:

$$\begin{cases} k_{1eq,i}k_{2eq,i} - \left(k_{1eq,i}m_2 + k_{2eq,i}m_2 + k_{2eq,i}m_1\right)\omega_{1-i} + m_1 m_2 \omega_{1,i}^2 = 0 \\ k_{1eq,i}k_{2eq,i} - \left(k_{1eq,i}m_2 + k_{2eq,i}m_2 + k_{2eq,i}m_1\right)\omega_{2,i} + m_1 m_2 \omega_{2,i}^2 = 0 \end{cases}, \quad \text{for } i = 1, 2, \cdots, m \tag{21}$$

By solving this equations simultaneously, we can obtain the equivalent stiffness parameters, $k_{1eq,i}, k_{2eq,i}$, for each window. Now, assuming the stiffness to be a polynomial, we can find these coefficients by solving the set of equations by means of least square method:

$$\begin{cases} k_{1eq,1} = k_1^I + 0.75k_1^{III}\left(X_{eq1,1}^{III}\right)^2 + 0.595k_1^V\left(X_{eq1,1}^V\right)^4 + 0.506k_1^{VII}\left(X_{eq1,1}^{VII}\right)^6 \\ k_{1eq,2} = k_1^I + 0.75k_1^{III}\left(X_{eq1,2}^{III}\right)^2 + 0.595k_1^V\left(X_{eq1,2}^V\right)^4 + 0.506k_1^{VII}\left(X_{eq1,2}^{VII}\right)^6 \\ \vdots \\ k_{1eq,m} = k_1^I + 0.75k_1^{III}\left(X_{eq1,m}^{III}\right)^2 + 0.595k_1^V\left(X_{eq1,m}^V\right)^4 + 0.506k_1^{VII}\left(X_{eq1,m}^{VII}\right)^6 \end{cases} \tag{22}$$

the similar set of equations can be solved for the second spring.

Conclusion

This paper developed a procedure to identify nonlinear stiffness in dynamical systems by utilizing wavelet transform on the system's free decay response. The concept of equivalent linearized stiffness is proposed in which the nonlinear stiffness of the system is replaced with an equivalent linear one which is function of oscillation amplitude. In the case of non-sinusoid oscillation, an equivalent amplitude is obtained; consequently, the eigenfrequencies of the nonlinear system is found as a function of time. Using the dominant scale concept, we extract the experimental eigenfrequencies of the system from the time-frequency representation of the system. Finally, a set of equations is intended to identify the polynomial coefficients of the nonlinear stiffness.

Reference

[1] Tomkinson G.R., *Linear or nonlinear – that is the question*, ISMA19, 11 (1994) 11-32.

[2] Newland, D.E., *Wavelet analysis of Vibration*, J sound & vibration 116 (1994) 409-425.

[3] Yu K. et al., *Missile flutter experiment and data analysis using wavelet transform*, J. of sound and vibration 269 (2004) 899-912.

[4] Gouttebroze S., Lardies J., *On using the wavelet transform in modal analysis*, Mechanics research communications 28 (2001) 561-569.

[5] Le T., Argoul P., *Continuous wavelet transform for modal identification using free decay response*, J. of sound and vibration 277 (2004) 73-100.

[6] Staszewski W.J., Chance J.E., *Identification of nonlinear systems using wavlets – experimental study*, IMAC97, 1012-1016

[7] Lind R., Synder K., Brenner M., *Wavelet analysis to charachterise non-linearities and predict limit cycles of an aeroelastic system*, mechanical systems and signal processing 15 2 (2001) 337-356.

[8] Bellizi S., Guillemain P., *Identification of coupled non-linear modes from free vibration using time-frequency representations*. J. of sound and vibration 243 (2001) 191-213.

[9] Daubechies, I., *Ten lectures on wavelet*, SIAM, 1992.