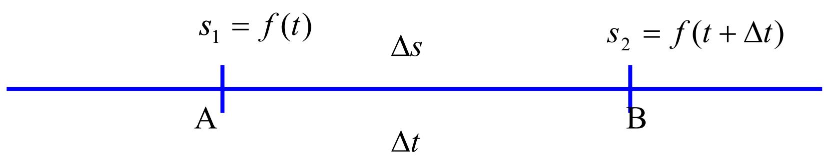
Consider a person moves on the following line with displacement $s = f(t)$, now consider the motion of this person between two specific points A and B.



The person cuts a displacement Δs between the two points A and B in a time Δt. The average velocity (v_{ave}) is given as:

$$v_{ave} = \frac{\Delta s}{\Delta t},$$

From the graph we can easily see that $\Delta s = f(t+\Delta t) - f(t)$, then

$$v_{ave} = \frac{f(t+\Delta t) - f(t)}{\Delta t},$$

i.e, v_{ave} is the average rate of change of the displacement with respect to t between the two points A and B (when the time changes by Δt).

For the equation $v_{ave} = \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$, take the limit of each side as Δt approaches 0, we obtain:

$$\lim_{\Delta t \to 0} v_{ave} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{df(t)}{dt} = \frac{ds}{dt},$$

The velocity $v = \lim_{\Delta t \to 0} v_{ave}$ is called the instantaneous velocity (i.e, the velocity of the person at any time t). Therefore:

$$v = f'(t) = \frac{ds}{dt}.$$

Example 1 the displacement of a function moving along a number line is given as: $s = f(t) = 3t^2 + 5$, where the time t is given in second and the displacement s is given in meters,

(i) find the average velocity over the interval $[10, 10.1]$,

(ii) find the velocity when t = 10.

Solution:

The velocity over the interval $[10,10.1]$ is the average velocity v_{ave} and it is calculated using the law $v_{ave} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$, now:

$\Delta t = 10.1 - 10 = 0.1$

$f(t+\Delta t) = f(10.1) = 3(10.1)^2 + 5 = 311.03,$

$f(t) = f(10.1) = 3(10)^2 + 5 = 305$, then

$v_{ave} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{311.03 - 305}{0.1} = \frac{6.03}{0.1} = 60.3\ m/\sec.$

To find the velocity at t = 10, this means that we need to determine the instantaneous velocity v , where $v = f'(t) = \frac{ds}{dt}$, this implies that $v = 6t$, and finally the velocity when t = 10 is:

$$v = 6(10) = 60 \quad m/\sec.$$

Remark, Notice the average velocity in this example is very close to the instantaneous velocity, because the length of time interval here is very small.

In general we have the following comparison between $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$:

$\frac{\Delta y}{\Delta x}$	$\frac{dy}{dx}$
It is the **average** rate of change of y with respect to x over the interval x to x + Δx .	It is the instantaneous rate of change of y with respect to x.

It is an important fact that $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, so when Δx is very small (close to zero) we may consider $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$, this implies that:

$$\Delta y \approx \frac{dy}{dx} \Delta x .$$

This relation gives the change in y as x changes by Δx . For instance, if x changes by 1, an estimate of the change in y is $\frac{dy}{dx}$.

Example 2 suppose that $y = f(x)$ and $\frac{dy}{dx} = 8$ when x = 3. Estimate the change in y if x changes from 3 to 3.5.

Solution:

$\Delta x = 3.5 - 3 = 0.5,$ using the relation $\Delta y \approx \frac{dy}{dx}\Delta x$, leads to $\Delta y \approx (0.8)(0.5) = 4.$ i.e, when x changes from 3 to 3.5 y changes by 4.

Example 3 find the rate of change of $y = x^4$ with respect to x, and evaluate it when x = 2 and when x = −1.

Solution:

The rate of change of f(x) is $\frac{dy}{dx} = 4x^3$,

The rate of change of y with respect to x at x = 2 is $\left.\frac{dy}{dx}\right|_{x=2} = 4(2)^3 = (4)(8) = 32.$

The rate of change of y with respect to x at x = −1 is $\left.\frac{dy}{dx}\right|_{x=-1} = 4(-1)^3 = (4)(-1) = -4.$

Example 4 let $p = 100 - q^2$ be the demand function for a manufacturer's product. Find the rate of change of price p per unit with respect to quantity q. How fast is the price changing with respect to q when q = 5? Assume that P is in Ryals.

Solution:

The rate of change of p with respect to q is $\frac{dp}{dq}$, then

$$\frac{dp}{dq} = \frac{d}{dq}(100 - q^2) = \frac{d}{dq}100 - \frac{d}{dq}q^2 = 0 - 2q = -2q\,.$$

At q = 5, then $\frac{dp}{dq} = -2(5) = -10$. This means that when the quantity is changed from 5 to 6 the price p will be changed by -10 Ryals per unit.

The total cost function is expressed by $c = f(q)$, where q is the marketing units by a manufacturer's and c is its corresponding cost. Thus, the marginal cost is defined as follows:

$$\text{Marginal cost} = \frac{dc}{dq}.$$

Also, the average cost per unit $\bar{c}$ is given as $\bar{c} = \frac{c}{q}$.

Example 5 if a manufacturer average cost is $\bar{c} = 0.1q^2 - 0.2q + 3 + \frac{1000}{q}$, then find the marginal cost function?. What is the marginal cost when 20 units are produced?

Solution:

Since $c = q\bar{c}$ then $c = q\bar{c} = q(0.1q^2 - 0.2q + 3 + \frac{1000}{q})$,

$\therefore\ c = 0.1q^3 - 0.2q^2 + 3q + 1000,$ Now

The marginal cost function

$$\frac{dc}{dq} = (0.1)(3)q^2 - (0.2)(2)q + 3$$
$$= 0.3q^2 - 0.4q + 3.$$

The marginal cost when 20 units are produced = $\left.\frac{dc}{dq}\right|_{q=20} = 0.3(20)^2 - 0.4(20) + 3 = 115.$ Thus, if c is in Ryals and production is increased by one unit from 20 to 21, then the cost of additional unit is approximately 115 Ryals.

If $r = f(q)$ is the total-revenue function for a manufacturer, then the marginal revenue is defined by:

$$\text{Marginal revenue} = \frac{dr}{dq}.$$

Marginal revenue measures the rate at which revenue changes with respect to units sold.

Also, $\dfrac{\frac{dr}{dq}}{r(q)}$ is called the <u>relative rate of change of revenue</u>, and

$\left(\dfrac{\frac{dr}{dq}}{r(q)}\right).(100)$ is the <u>percentage rate of change</u> of the revenue.

Example (6) if the total revenue is given as $r = 2q$, find:

(i) the marginal revenue,

(ii) the relative rate of change of revenue when q = 50,

(iii) the percentage rate of change of revenue when q = 50.

Solution:

The marginal revenue = $\dfrac{dr}{dq} = \dfrac{d}{dq}(2q) = 2$,

The marginal revenue when q = 50 is $\left.\dfrac{dr}{dq}\right|_{q=50} = 2$.

The total revenue when q = 50 is $r = 2(50) = 100$, the relative rate of change of revenue

when q = 50 is $\dfrac{\frac{dr}{dq}}{r(q)} = \dfrac{2}{100} = 0.02$.

The percentage rate of change of revenue when q = 50 is then equals (0.02).(100) = 2%.

Home work: Solve book pages 561, 562 and 563, the following problems:

[8] A position function is given as $s = 3t^4 - t^{7/2}$, along the time interval $[0, \frac{1}{4}]$, (a) find the position at t = 0, (b) find the average velocity over the given interval, (c) find the velocity at t = 0.

[12] The volume of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Find the rate of change of volume with respect to radius when $r = 6.3 \times 10^{-4}$ cm.

[16] if the cost function c of producing q units of a product is given as follows: $c = 0.1q^2 + 3q + 2$, then find the marginal cost function?. What is the marginal cost when 3 units are produced?

[20] If a manufacturer average cost is $\bar{c} = 2 + \frac{1000}{q}$, then find the marginal cost function?. What is the marginal cost when q = 25 and q = 235?.

[39] For the cost function $c = 0.2q^2 + 1.2q + 4$ how fast does c changes with respect to q when q = 5?. Determine the percentage rate of change of c with respect to q when q = 5.

[41] For a certain manufacturer, the revenue obtained from the sale of q units of a product is given by :

$$r = 30q - 0.3q^2,$$

(a) How fast does r change with respect to q? when q = 10,
(b) Find the relative rate of change of r,
(c) Find the percentage rate of change of the revenue r (approximate the result to the nearest unit).