First recall that if u is a function of x (i.e. u = u(x)), then the relation between the differential of u and the differential of x is given by:

$$du = u'(x)dx$$

Example(1) Find $I = \int (x+5)^7 dx$

Solution

Let $u = x+5$ $\Rightarrow du = dx$, and therefore $I = \int u^7 du = \frac{u^8}{8} + c$,

i.e. $I = \frac{(x+5)^8}{8} + c$.

Example(2) Find $I = \int 2x(x^2+3)^5 dx$

Solution

Let $u = x^2 + 3$ $\Rightarrow du = 2xdx$, and therefore $I = \int u^5 du = \frac{u^6}{6} + c$,

i.e. $I = \frac{(x^2+3)^6}{6} + c$.

Rule (1):

The integral $\int u'(x).[u(x)]^n dx = \frac{[u(x)]^{n+1}}{n+1} + c$. i.e. The integral of a bracket (power n) multiplied by the derivative of what inside the bracket = $\frac{(the\ bracket)^{n+1}}{n+1} + c$

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In the last example we observe that, the function inside the bracket $x^2 + 3$ is multiplied by its derivative $2x$, therefore we integrate the function inside the bracket directly, i.e.

$$I = \int 2x(x^2+3)^5 dx = \frac{(x^2+3)^6}{6} + c.$$

Example(3) Find $I = \int x^2 (3x^3 + 7)^3 dx$

Solution:

Since $9x^2$ is the derivative of the function $(3x^3+7)^3$, then we multiply and divide the integral I by 9.

$$\therefore \quad I = \frac{1}{9}\int 9x^2 (3x^3+7)^3 dx = \frac{1}{9}.\frac{(3x^3+7)^4}{4} + c = \frac{(3x^3+7)^4}{36} + c \,.$$

Example(4) Find $I = \int (7x-6)^4 dx$

Solution:

Since 7 is the derivative of the function ($7x-6$), then we multiply and divide the integral I by 7.

$$\therefore \quad I = \frac{1}{7}\int (7x-6)^4 dx = \frac{1}{7}.\frac{(7x-6)^5}{5} + c = \frac{(7x-6)^5}{35} + c \,.$$

Rule (2):

The integral $\int u'(x).e^{u(x)} dx = e^{u(x)} + c$. i.e. The integral of e power function multiplied by the derivative of the function = the exponential function plus constant.

Proof : $I = \int u'(x).e^{u(x)} dx$

Let $y = e^{u(x)}$ $\quad dy = u'(x).e^{u(x)} dx$ then $I = \int dy = y + c$, i.e.

$\int u'(x).e^{u(x)} dx = e^{u(x)} + c$.

Example(5) Find $I = \int 3.e^{3x} dx$

Solution:

Since (3) is the derivative of the function ($3x$):

$I = \int 3.e^{3x} dx = e^{3x} + c$.

Example(6) Find $I = \int (t+1).e^{t^2+2t}\,dt$

Solution:

Since 2(t + 1) is the derivative of the function ($t^2 + 2t$),then we multiply and divide the integral I by 2

$$I = \frac{1}{2}\int 2(t+1).e^{t^2+2t}\,dt = \frac{1}{2}e^{t^2+2t} + c\,.$$

Since $\frac{d}{dx}\ln|x| = \frac{1}{x}$, then

$$\int \frac{1}{x}dx = \ln|x| + c$$

In general we have the following rule:

Rule (3):

The integral $\int \frac{u'(x)}{u(x)}dx = \ln|u(x)| + c$. i.e. when the numerator becomes a derivative of the denominator, then their integral = ln(denominator) plus constant.

Proof:

$$I = \int \frac{u'(x)}{u(x)}dx$$

Let $y = \ln|u(x)| \quad \Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\ln|u(x)|$

Since $\frac{d}{dx}\ln|u(x)| = \frac{u'(x)}{u(x)}$, then $d\ln|u(x)| = \frac{u'(x)}{u(x)}.dx$

Now, $I = \int \frac{u'(x)}{u(x)}dx = \int dy = y + c$, this implies that

$$I = \int \frac{u'(x)}{u(x)}dx = \ln|u(x)| + c\,.$$

Example(7) Find $I = \int \frac{1}{x+5} dx$

Solution:

Since the numerator (1) is a derivative of the denominator ($x+5$), then

$$I = \int \frac{1}{x+5} dx = \ln|x+5| + c.$$

Example(8) Find $I = \int \frac{3x^2+4x^3}{x^3+x^4} dx$

Solution:

Since the numerator ($3x^2+4x^3$) is a derivative of the denominator (x^3+x^4), then

$$I = \int \frac{3x^2+4x^3}{x^3+x^4} dx = \ln|x^3+x^4| + c.$$

Example(9) Find $I = \int \frac{2x^3+3x}{x^4+3x^2+7} dx$

Solution:

Since 2($2x^3+3x$) is a derivative of the denominator (x^4+3x^2+7), then we multiply and divide the integral I by 2

$$I = \frac{1}{2}\int 2.\frac{2x^3+3x}{x^4+3x^2+7} dx = \frac{1}{2}\ln|x^4+3x^2+7| + c$$

$$= \ln|x^4+3x^2+7|^{\frac{1}{2}} + c = \ln\sqrt{x^4+3x^2+7} + c.$$

Example(10) Find $I = \int [\frac{x}{x^2+1} + \frac{x^5}{(x^6+1)^2}] dx$

Solution

$$I = \int [\frac{x}{x^2+1} + \frac{x^5}{(x^6+1)^2}] dx = \int \frac{x}{x^2+1} dx + \int \frac{x^5}{(x^6+1)^2} dx$$

Let $I_1 = \int \frac{x}{x^2+1} dx$ and $I_2 = \int \frac{x^5}{(x^6+1)^2} dx$, then

$I = I_1 + I_2$,

We first try to find $I_1 = \int \frac{x}{x^2+1}dx$

Since ($2x$) is a derivative of the denominator (x^2+1), then we multiply and divide the integral I_1 by 2

$$I_1 = \frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\ln\left|x^2+1\right| + c_1 .$$

Second we calculate $I_2 = \int \frac{x^5}{(x^6+1)^2}dx = \int x^5(x^6+1)^{-2}dx$

Since ($6x^5$) is a derivative of the function (x^6+1), then we multiply and divide the integral I_2 by 6

$$I_2 = \frac{1}{6}\int 6x^5(x^6+1)^{-2}dx = \frac{1}{6}\frac{(x^6+1)^{-2+1}}{-2+1} + c_2 = \frac{1}{6}\frac{(x^6+1)^{-1}}{-1} + c_2 = -\frac{(x^6+1)^{-1}}{6} + c_2$$

Now, since $I = I_1 + I_2$, then

$$I = \frac{1}{2}\ln\left|x^2+1\right| - \frac{(x^6+1)^{-1}}{6} + c.$$

Home work: Solve exercises 14.4, problems: 9, 15, 35, 53, 70, 75.