Rule (1) The derivative of constant function is 0,

$$\frac{d}{dx}c = 0$$

Proof

$f(x) = c$ and $f(x+h) = c$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

Rule (2) The power rule, if n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

Proof

$f(x) = x^n$ and $f(x+h) = (x+h)^n$, then

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ... + nxh^{n-1} + h^n] - x^n}{h}$$

$$= \lim_{h\to 0}\frac{h[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + ... + nxh^{n-2} + h^{n-1}]}{h}.$$

$$= nx^{n-1} + 0 + ... + 0 + 0$$

$$= nx^{n-1}.$$

Example (1) Find the derivative of the following functions:

$(i)\ f(x) = x^5,$ $(ii)\ f(x) = x,$ $(iii)\ f(t) = t^6.$

Solution

(i) $\frac{df(x)}{dx} = f'(x) = 5x^{5-1} = 5x^4.$

(ii) $\frac{df(x)}{dx} = \frac{d}{dx}x = (1)x^{1-1} = (1)x^0 = (1).(1) = 1.$

(iii) $\frac{df(t)}{dt} = f'(t) = \frac{d}{dt}t^6 = 6t^{6-1} = 6t^5.$

A generalization for the power rule is given as follows:

For any real number n

$$\frac{d}{dx}x^n = nx^{n-1}$$

Example (2) If $f(x)=\sqrt{x}$, find $f'(x)$

Solution

$f(x)=x^{\frac{1}{2}}$, then $f'(x)=\frac{df(x)}{dx}=\frac{d}{dx}x^{\frac{1}{2}}=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2}\frac{1}{x^{\frac{1}{2}}}=\frac{1}{2\sqrt{x}}$.

In other words, we say that, the derivative of the square root = $\frac{1}{2.\ square\ \ root}$

Example (3) Find $\frac{dp}{dq}$ if $p=\sqrt[3]{q^2}$

Solution

$$\frac{dp}{dq}=\frac{d}{dq}q^{\frac{2}{3}}=\frac{2}{3}q^{\frac{2}{3}-1}=\frac{2}{3}q^{-\frac{1}{3}}=\frac{2}{3q^{\frac{1}{3}}}=\frac{2}{3\sqrt[3]{q}}.$$

Example (4) Find the derivative of the following function: $f(x)=\frac{1}{x}$,

Solution $f(x)=\frac{1}{x}=x^{-1}$, then $f'(x)=\frac{d}{dx}x^{-1}=-x^{-2}=-\frac{1}{x^2}$.

Example (5) Find an equation of the tangent line to the curve $y=x\sqrt{x}$ at the point $(4,8)$

Solution $y=x\sqrt{x}=x(x)^{\frac{1}{2}}=x^{\frac{3}{2}}$,

Thus, $\frac{dy}{dx}=\frac{d}{dx}x^{\frac{3}{2}}=\frac{3}{2}x^{\frac{3}{2}-1}=\frac{3}{2}x^{\frac{1}{2}}=\frac{3}{2}\sqrt{x}$,

We know that the derivative of a function at a point p is the slope m of the tangent line at this point p,

$$\therefore\ m=\left.\frac{dy}{dx}\right|_{p(4,8)}=\left.\frac{dy}{dx}\right|_{x=4}=\frac{3}{2}\sqrt{4}=3,$$

$$\therefore\ m=\frac{y-8}{x-4}=3,$$

$\Rightarrow$ $y-8=3(x-4)$, thus, the equation is: $y=3x-4$.

Rule (3) The derivative of a constant by a function = the constant multiplied by the derivative of the function. i.e., $\frac{d}{dx}[c.f(x)] = c.\frac{d}{dx}f(x)$.

Proof:

Let $g(x) = cf(x)$ then $g(x+h) = cf(x+h)$,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$
$$= \lim_{h\to 0}\frac{c[f(x+h)-f(x)]}{h} = c.\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c.f'(x).$$

Example (6) Find:

(i) $\frac{d}{dx}2x^5 = 2\frac{d}{dx}x^5 = (2)(5)x^4 = 10x^4$.

(ii) $\frac{d}{dx}(-\frac{1}{3}x^3) = -\frac{1}{3}.\frac{d}{dx}(x^3) = -\frac{1}{3}(3)x^2 = -x^2$.

Rule (4) The derivative of sum of two functions equals sum of their derivatives, i.e.

$$\frac{d}{dx}[f(x)+g(x)] = \frac{df(x)}{dx} + \frac{dg(x)}{dx}.$$

Also, the derivative of difference of two functions equals difference of their derivatives, i.e.

$$\frac{d}{dx}[f(x)-g(x)] = \frac{df(x)}{dx} - \frac{dg(x)}{dx}.$$

In general

$$\frac{d}{dx}[f(x)\pm g(x)\pm ....\pm h(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx} \pm .... \pm \frac{dh(x)}{dx}.$$

Example (7)

$$\frac{d}{dx}[3x^{10}+4x^5] = 3\frac{d}{dx}x^{10} + 4\frac{d}{dx}x^5$$
$$= 3(10)x^9 + 4(5)x^4$$
$$= 30x^9 + 20x^4.$$

Example (8) Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

Solution

$\frac{dy}{dx} = 6x^2 + 6x - 12$.

If the tangent line is horizontal, then its slope m = 0, thus

$\Rightarrow \quad 6x^2 + 6x - 12 = 0$

$\Rightarrow \quad x^2 + x - 2 = 0 \quad \Rightarrow \quad x = 1 \quad or \quad x = -2$,

At x = 1 then $y = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$, this implies that one of the points is $(1,-6)$.

At $x = -2$ then $y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 21$, this implies that the other point is $(-2,21)$.

Thus, the points at which the tangent line is horizontal are $(1,-6)$ and $(-2,21)$.

Example (9) Where does the normal line to the parabola $y = x - x^2$ at the point $(1,0)$ intersect the parabola a second time?

Solution

The slope of the tangent line to the parabola at the point (x, y) is $m = y' = 1 - 2x$. Therefore, the slope the tangent line to the parabola at the point $(1,0)$ is $m_1 = \left.\frac{dy}{dx}\right|_{x=1} = 1 - 2(1) = -1$. However, the normal line to the tangent line at $(1,0)$ has a slope m_2 and given from the relation:

$$m_2 = -\frac{1}{m_1}.$$

$\Rightarrow \quad m_2 = -\frac{1}{-1} = 1$. Since this normal line passes through the point $(1,0)$, then its equation is given from:

$\therefore \; m_2 = \frac{y - 0}{x - 1} = 1$. i.e. the equation of the normal line at $(1,0)$ is $y = x - 1$.

Now, the normal line intersects the parabola when

$$x - 1 = x - x^2,$$

$\Rightarrow \; x^2 = 1 \qquad \Rightarrow x = \pm 1$. Thus the intersection points are: $(1,0)$ and $(-1,-2)$.

Example (10)

Let
$$f(x)=\begin{cases}2-x & if\ \ x\le 1\\ x^2-2x+2 & if\ \ x>1.\end{cases}$$

Is the function $f(x)$ differentiable at $x=1$?

Solution

From the definition of the function $f(x)$ and using the laws of derivation it follows that:

$$f'(x)=\begin{cases}-1 & if\ \ x\le 1\\ 2x-2 & if\ \ x>1.\end{cases}$$

Now,

$f'(1^+)=\lim\limits_{x\to 1^+}\dfrac{f'(x)-f'(1)}{x-1}=2(1)-2=0$, and $f'(1^-)=\lim\limits_{x\to 1^-}\dfrac{f'(x)-f'(1)}{x-1}=-1$.

Thus, $f'(1^+)\ne f'(1^-)$ and therefore, $f'(1)$ doesn't exist. i.e. the function $f(x)$ is **not** differentiable at $x=1$.

Derivative of exponential functions:

Our aim is to find the derivative of $f(x)=e^x$. First we introduce the following definition:

Definition: The number e is a number such that $\lim\limits_{h\to 0}\dfrac{e^h-1}{h}=1$.

Then $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{e^{x+h}-e^x}{h}=\lim\limits_{h\to 0}\dfrac{e^x.[e^h-1]}{h}=e^x\lim\limits_{h\to 0}\dfrac{[e^h-1]}{h}=e^x.(1)=e^x$,

Thus,
$$\frac{d}{dx}e^x=e^x.$$

Example (11) If $f(x)=e^x-5x$, find $f'(x)$.

Solution

$$f'(x)=\frac{df(x)}{dx}=\frac{d}{dx}(e^x-5x)=\frac{d}{dx}e^x-\frac{d}{dx}5x=e^x-5\frac{d}{dx}x=e^x-5(1)$$
$$=e^x-5.$$