We can write the derivative as a function as follows:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example (1) Find the derivative of the function $f(x)=\frac{1}{x^2}$ using the definition. State the domain of the function and its derivative.

Solution

$$f'(x) = \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{x^2-(x+h)^2}{x^2(x+h)^2} = \lim_{h\to 0}\frac{1}{h}\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{-2xh-h^2}{x^2(x+h)^2} = \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2} = \frac{-2x}{x^2(x+0)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}.$$

The domain of f and f' is $R-\{0\}$.

Example (2) If $f(x)=x-\frac{2}{x}$, find $f'(x)$ using the definition of the derivative.

Solution

Since $f(x)=x-\frac{2}{x}$, then $f(x+h)=x+h-\frac{2}{x+h}$

$$\therefore\ f'(x) = \lim_{h\to 0}\frac{1}{h}\{x+h-\frac{2}{x+h}-(x-\frac{2}{x})\}$$

$$= \lim_{h\to 0}\frac{1}{h}\{(x+h-x)-(\frac{2}{x+h}-\frac{2}{x})\}$$

$$= \lim_{h\to 0}\frac{1}{h}\{(h)-2(\frac{1}{x+h}-\frac{1}{x})\} = \lim_{h\to 0}1-2\lim_{h\to 0}\frac{1}{h}(\frac{x-x-h}{x(x+h)} = 1-2\lim_{h\to 0}\frac{1}{h}\{\frac{-h}{x(x+h)}\}$$

$$= 1+\frac{2}{x^2}.$$

We may write the derivative using other symbols as follows:

$$y' \quad or \quad \frac{dy}{dx} \quad or \quad D_x f(x) \quad or \quad Df(x),$$

Also,

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

Definition The function $f(x)$ is said to be **differentiable** at the point a if

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

exists. If the limit doesn't exist then f(x) is not differentiable.

Corollary The polynomial function is differentiable at any point in its domain.

Theorem Every differentiable function is continuous.

Proof:

Let the function $f(x)$ be differentiable at a point $(a, f(a))$, then $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists and equal a real number, that is $f'(a)$.

Now,

$$f(a+h)-f(a)=\frac{f(a+h)-f(a)}{h}.h,$$

By taking the limit in both sides as h approaches to zero, then

$$\begin{aligned}\lim_{h\to 0}[f(a+h)-f(a)] &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}h\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.\lim_{h\to 0}h\\ &= f'(a).0\\ &= 0.\end{aligned}$$

This implies that

$$\lim_{h\to 0}[f(a+h)-f(a)] = \lim_{h\to 0}f(a+h)-\lim_{h\to 0}f(a)=\lim_{h\to 0}f(a+h)-f(a)=0.$$

Consequently,

$$\lim_{h\to 0}f(a+h)=f(a).$$

By putting $x=a+h$, then $x\to a$ as $h\to 0$, and the last limit can be rewritten as:

$$\lim_{x\to a}f(x)=f(a).$$

This implies that the function $f(x)$ is continuous at the point a.

The converse of the above theorem is not true. i.e. **not** every contiuous function is differentiable.

The following example shows that not every every contiuous function is differentiable.

Example (3) Show that the function $f(x)=|x|$ is not differentiable at $x=0$.

Solution

The function $f(x)=|x|$ is <u>continuous</u> at $x=0$ (why?)

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|0+h|-|0|}{h}=\lim_{h\to 0}\frac{|h|}{h}.$$

Since $\frac{|h|}{h}=\begin{cases}1 & at\ h>0\\ -1 & at\ h<0\end{cases}$

Then,

$$\lim_{h\to 0^+}\frac{|h|}{h}=1 \qquad \text{and} \qquad \lim_{h\to 0^-}\frac{|h|}{h}=-1.$$

This implies that, $\lim_{h\to 0}\frac{|h|}{h}$ doesn't exist and consequently $f'(a)$ doesn't exist. i.e. the function $f(x)=|x|$ is not differentiable at $x=0$.

Example (4) Show that the function $f(x)=|x-6|$ is not differentiable at $x=6$.

Solution

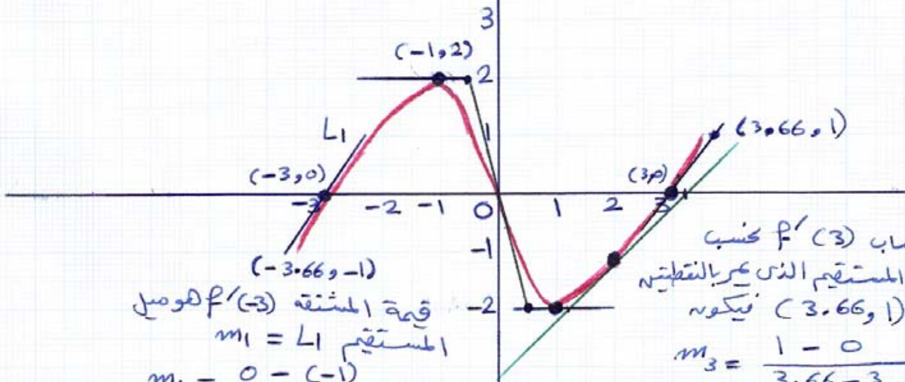
$$f'(6)=\lim_{h\to 0}\frac{f(6+h)-f(6)}{h}=\lim_{h\to 0}\frac{|6+h-6|-|6-6|}{h}=\lim_{h\to 0}\frac{|h|}{h}$$

We have already proved that $\lim_{h\to 0}\frac{|h|}{h}$ doesn't exist from the last example. This means that $f'(6)$ doesn't exist. i.e. the function $f(x)=|x-6|$ is not differentiable at $x=6$.

Page 173 (EXC. 2.9) problem number 3

Exercises 2.9 page 173, PNo.(3)

لإيجاد f' عند كل نقطه نرسم المماسات عند كل نقطة بالمسطرة ، ثم نحدد احداثيات نقطتين على الخط المستقيم المرسوم ويكون ميل هذا الخط هو قيمة المشتقة عند النقطة المرسوم عندها المماس



قيمة المشتقة $f'(-3)$ هو ميل المستقيم $m_1 = L_1$

$$m_1 = \frac{0-(-1)}{-3-(-3.66)} = \frac{+1}{0.66} \simeq 1.51$$

$\therefore f'(3) \simeq 1.51$

بالمثل لحساب $f'(3)$ نحسب ميل الخط المستقيم الذى يمر بالنقطتين $(3,0)$ و $(3.66,1)$ فيكون

$$m_3 = \frac{1-0}{3.66-3} = \frac{1}{0.66} \approx 1.51$$

$\therefore f'(3) \simeq 1.51$

similarly

$$f'(2) = m_4 = \frac{0-(-2)}{3.1-1.1} = \frac{2}{2}$$

$\therefore f'(2) = 1$

حيث أن m_4 هو ميل المماس عند النقطه $x=2$ والذى يمر بالنقطتين $(1.1,-2)$ and $(3.1,0)$

* من الواضح أن المماس عند كلا النقطتين $(-1,2)$ ، $(1,2)$ أفقى horizontal وبالتالى يكون ميل المماس عند كلا النقطتين صفر ، أى أن

$f'(1) = 0$ and $f'(-1) = 0$.

* المماس الذى يمر بالنقطه $(0,0)$ يمر أيضاً بالنقطه $(-0.5,2)$ وبالتالى يكون ميله

$$m_2 = \frac{0-2}{0-(-0.5)} = \frac{-2}{0.5} = -4$$

$\therefore f'(0) = -4$