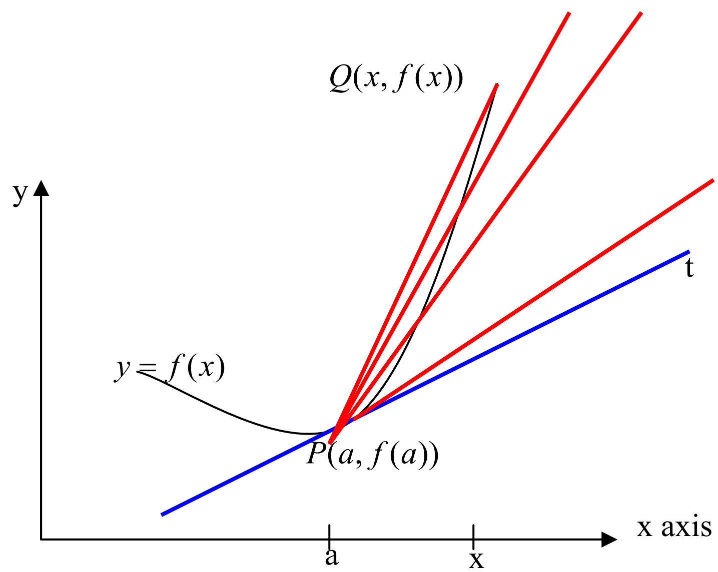


Consider the point P on the curve of the function f(x) with coordinate $(a, f(a))$ and the point Q is sufficiently small increasement of the point P with coordinates $(x, f(x))$.

The slope of the secant line PQ is m_{PQ} and given by:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}. \qquad (1)$$

When the point Q moves on the curve of f(x) to reach P, the secant lines (red lines) will move to coincide with the tangent line t-blue line- of the point P (this only happens as x approaches to the point a).

Thus, the limiting behavior of the slope of the secant lines approaches to the slope of the curve at the point x as a.

Example (1) find an equation of the tangent line to the parabola $y = x^2$ at the point P(1, 1).

Solution since a = 1, f(a) = 1

Then $m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1}(x+1) = 1 + 1 = 2,$

Since m = $\frac{\textit{difference in y coordinates}}{\textit{difference in x coordinates}}$, then $\frac{y-1}{x-1}=2$,

This implies that $y-1=2(x-1)$, and the equation of the tangent line of P is then given by

$$y=2x-1.$$

Equation (1) can be rewritten in a simpler form if h represents the distance from a to x then $x=a+h$. Thus as $x\to a$ then $h\to 0$. Consequently equation (1) can be rewritten as

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}. \qquad (2)$$

Example (2) find an equation of the tangent line to the curve $y=\frac{3}{x}$ at the point (3, 1).

Solution since a = 3, f(a) = 1 and

$f(a+h)=f(3+h)=\frac{3}{3+h}$, now using equation (2), we obtain:

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\frac{3}{3+h}-1}{h}=\lim_{h\to 0}\frac{\frac{3-(3+h)}{3+h}}{h}=\lim_{h\to 0}\frac{\frac{-h}{3+h}}{h}=\lim_{h\to 0}\frac{-1}{3+h}=-\frac{1}{3}$$

Since m = $\frac{\textit{difference in y coordinates}}{\textit{difference in x coordinates}}$, then $\frac{y-1}{x-3}=-\frac{1}{3}$,

and the equation of the tangent line of P (3, 1) is then given by $y=-\frac{1}{3}x+2$.

Example (3) find an equation of the tangent line to the curve $y=2x/(x+1)^2$ at the point P(0, 0).

Solution since a = 0, f(a) = 0

Then $m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to 0}\frac{x^2-1}{x-1}=\lim_{x\to 0}\frac{\frac{2x}{(x+1)^2}-0}{x}=\lim_{x\to 0}\frac{2}{(x+1)^2}=\frac{2}{(0+1)^2}=2,$

Since m = $\frac{\textit{difference in y coordinates}}{\textit{difference in x coordinates}}$, then $\frac{y-0}{x-0}=2$,

Then the equation of the tangent line of P (0, 0) is $y=2x$.

Example (4) (a) find the slope of the tangent line to the parabola $y = 1 + x + x^2$ at the point where x = a. (b) find the slopes of the tangent lines at the points whose x-coordinates are

$(i)\ -1,\quad (ii)\ -\frac{1}{2},\quad$ (iii) 1.

Solution

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{1 + x + x^2 - (1 + a + a^2)}{x - a} = \lim_{x \to a} \frac{x + x^2 - a - a^2}{x - a} = \lim_{x \to a} \frac{(x - a) + (x^2 - a^2)}{x - a}$$

$$= \lim_{x \to a} \frac{(x - a) + (x - a)(x + a)}{x - a} = \lim_{x \to a} \frac{(x - a)\{1 + (x + a)\}}{(x - a)} = \lim_{x \to a} \{1 + (x + a)\} = \{1 + (a + a)\} = 1 + 2a.$$

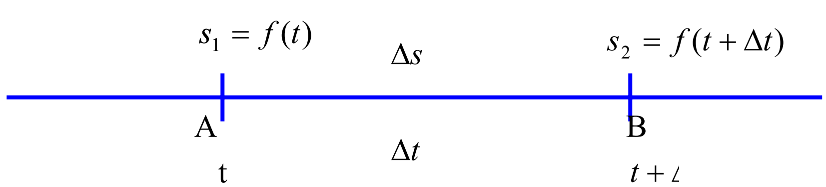
The slope as $a = -1$ is $1 + 2(-1) = 1 - 2 = -1$.

The slope as $a = -\frac{1}{2}$ is $1 + 2(-\frac{1}{2}) = 1 - 1 = 0$.

The slope as $a = 1$ is $1 + 2(1) = 1 + 2 = 3$.

Velocity:

Consider a person moves on the following line with displacement $s = f(t)$, now consider the motion of this person between two specific points A and B.



The person cuts a displacement Δs between the two points A and B in a time Δt. The <u>average velocity</u> (v_{ave}) is given as:

$$v_{ave} = \frac{\Delta s}{\Delta t},$$

From the graph we can easily see that $\Delta s = f(t+\Delta t) - f(t)$, then

$$v_{ave} = \frac{f(t+\Delta t) - f(t)}{\Delta t},$$

i.e, v_{ave} is the average rate of change of the displacement with respect to t between the two points A and B (when the time changes by Δt).

However, if the length of this time interval Δt is very small then, the average velocity will be approximately equal to the velocity at the point A (i.e. v_{ave} approaches to the velocity at the time t), the velocity at any time t is called <u>instantaneous velocity</u> and its symbol is v . This means that:

$$v = \lim_{\Delta t \to 0} v_{ave}$$

Thus,

$$v = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

If we consider the time t at the point A to be t = a, and rename Δt by h, the equation of the instantaneous velocity can be rewritten as

$$v = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

It is clear that right hand side of the above equation is the same of the right hand side of equation (2), this means that ***the velocity at a time t = a equals the slope of the tangent line to the curve*** $s = f(t)$ ***at the point P*** $(a, f(a))$.

Example (5) the displacement of a function moving along a number line is given as: $s = f(t) = 3t^2 + 5$, where the time t is given in second and the displacement s is given in meters,

(i) find the average velocity over the interval [10,10.1],

(ii) find the velocity when t = 10.

Solution:

The velocity over the interval [10,10.1] is the average velocity v_{ave} and it is calculated using the law $v_{ave} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$, now:

$\Delta t = 10.1 - 10 = 0.1$

$f(t+\Delta t) = f(10.1) = 3(10.1)^2 + 5 = 311.03,$

$f(t) = f(10.1) = 3(10)^2 + 5 = 305$, then

$$v_{ave} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{311.03 - 305}{0.1} = \frac{6.03}{0.1} = 60.3 \ m/\sec.$$

To find the velocity at t = 10, this means that we need to determine the instantaneous velocity v , such that $v = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$, where $f(a) = 3a^2 + 5$ and $f(a+h) = 3(a+h)^2 + 5$, then

$$v = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{3(a+h)^2 + 5 - (3a^2 + 5)}{h} = \lim_{h\to 0} \frac{3(a^2 + 2ah + h^2) + 5 - (3a^2 + 5)}{h}$$

$$= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 + 5 - 3a^2 - 5}{h} = \lim_{h\to 0} \frac{6ah + 3h^2}{h} = \lim_{h\to 0} \frac{h(6a + 3h)}{h} = \lim_{h\to 0}(6a + 3h) = 6a.$$

this implies that $v = 6t$, and finally the velocity when t = 10 is:

$$v = 6(10) = 60 \quad m/\sec.$$

Remark, Notice the average velocity in this example is very close to the instantaneous velocity, because the length of time interval here is very small.

Example (6) if a ball is dropped from the top of a tower whose height 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution first, we must state that when a body is fall down under the force of gravity, the its equation of motion $s = f(t) = \frac{1}{2}gt^2$, where $g = 9.8 \ m/\sec^2$ (g is the acceleration of earth gravity). In the following we will calculate the instantaneous velocity of this ball at any time t = a. which is given by the equation $v = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$, where $f(a) = \frac{1}{2}ga^2$ and $f(a+h) = \frac{1}{2}g(a+h)^2$, then

$$v = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}g(a+h)^2 - \frac{1}{2}ga^2}{h} = \frac{1}{2}\lim_{h\to 0} \frac{g(a+h)^2 - ga^2}{h} = \frac{1}{2}\lim_{h\to 0} \frac{g(a^2 + 2ah + h^2) - ga^2}{h}$$

$$= \frac{1}{2}\lim_{h\to 0} \frac{ga^2 + 2gah + gh^2 - ga^2}{h} = \frac{1}{2}\lim_{h\to 0} \frac{2gah + gh^2}{h} = \frac{1}{2}\lim_{h\to 0} \frac{h(2ga + gh)}{h} = \frac{1}{2}\lim_{h\to 0}(2ga + gh) = \frac{1}{2}(2ga) = ga.$$

The velocity of the ball after 5 seconds is $v = g.a = g.(5) = 5g \ m/\sec.$

Now, we will calculate the velocity of the ball as it reaches the ground. As the ball reached the ground it is then cut a distance 450 m, the time of this fall is given from:

$$s(t) = \frac{1}{2}gt = 450,$$

This implies that $t^2 = \frac{900}{g}$ and this leads to $t = \sqrt{\frac{900}{g}}$ second. Now the velocity as the ball reaches the ground is $v = ga = g\sqrt{\frac{900}{g}} = 9.8\sqrt{\frac{900}{9.8}} \approx 94\, m/\sec$.

Other rates of change:

In general for the function $y = f(x)$:

Δx is the change in x, i.e.

$$\Delta x = x_2 - x_1,$$

Δy is the change in y and is given from:

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1),$$

Now, $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called <u>the average rate of change of y with respect to x over the interval</u> $[x_1, x_2]$.

As $\Delta x \to 0$, then $x_2 \to x_1$. Hence, the instantaneous rate of change of y with respect to x is given by:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The right hand side of last equation is the slope of the tangent line of the curve $y = f(x)$ at the point $(x_1, f(x_1))$. Thus, the instantaneous rate of change of y with respect to x ($\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$) is interpreted as the slope of the tangent line of the curve $y = f(x)$ at the point $(x_1, f(x_1))$.

Example (7) The number N (in thousands) of cellular phone subscribes in Malaysia is shown in the following table

Year	1993	1994	1995	1996	1997
N	304	572	873	1513	2461

(a) Find the average rate of growth in each of the following cases:

(i) from 1995 to 1997

(ii) from 1995 to 1996

(iii) from 1994 to 1995

(b) Estimate the instantaneous rate of growth in 1995 by taking the average of two average rates of change.

(c) Estimate the instantaneous rate of growth in 1995 by measuring the slope of a tangent.

Solution

The average rate of change is given by $\frac{\Delta N(t)}{\Delta t}$, where t represents the year and N represents the number (in thousands) of cellular phone. Thus,

The average rate of growth from 1995 to 1997 is:

$$\frac{2461-873}{1997-1995}=\frac{2461-873}{2}=794 \text{ thousand/year.}$$

The average rate of growth from 1995 to 1996 is:

$$\frac{1513-873}{1996-1995}=\frac{1513-873}{1}=640 \text{ thousand/year.}$$

The average rate of growth from 1994 to 1995 is:

$$\frac{873-572}{1995-1994}=\frac{873-572}{1}=301 \text{ thousand/year.}$$

(c) the instantaneous rate of growth in 1995 is

$$r = \lim_{h \to 1995} \frac{f(1995+h) - f(1995)}{h}$$

Thus, to find the instantaneous rate when h = 1, we obtain:

$r_1 = \lim_{h \to 1995} \frac{f(1996) - f(1995)}{1}$, this implies that $r_1 \approx \frac{f(1996) - f(1995)}{1} = \frac{1513 - 873}{1} = 640$.

when $h = -1$, we obtain:

$r_2 = \lim_{h \to 1995} \frac{f(1995-1) - f(1995)}{-1} = \frac{f(1994) - f(1995)}{-1}$, this implies that

$r_2 \approx \frac{f(1994) - f(1995)}{-1} = \frac{572 - 873}{-1} = 301$.

Now,the average of the two average rates of change is: $(r_1 + r_2)/2 = (640 + 301)/2 = 470.5$ th/y

(d) Hint: Draw the data that given in the table, join the points by a smooth curve. Then draw tangent line to the curve at the point $(1995,873)$. Thus, the slope of the tangent line at this point is **427.5** which equals the instantaneous rate of growth in 1995.