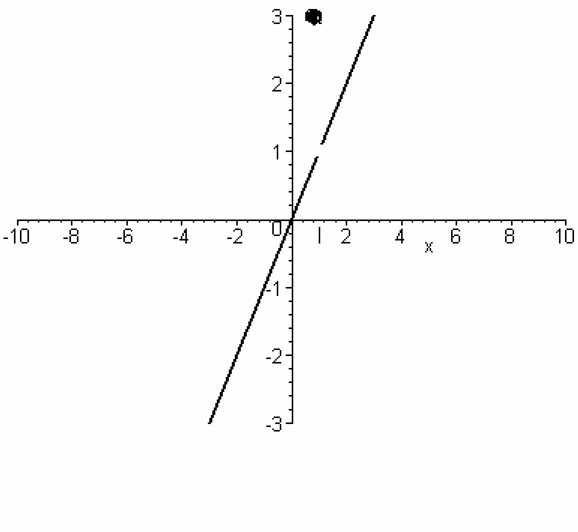
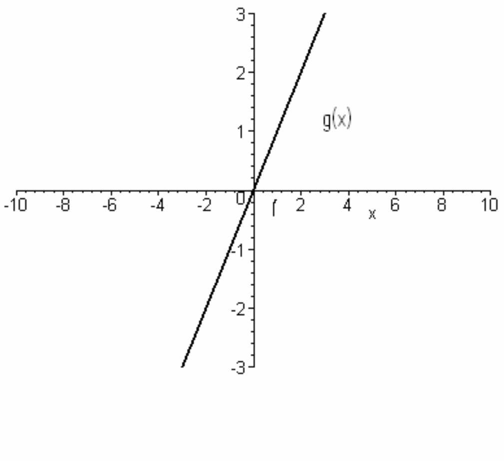
Continuity

Example (1) The function f(x) is defined as follows:

$$f(x)=\begin{cases} x & \text{if } \ x\neq 1, \\ 3 & \text{if } \ x=1. \end{cases}$$

By sketching the graph of this function:



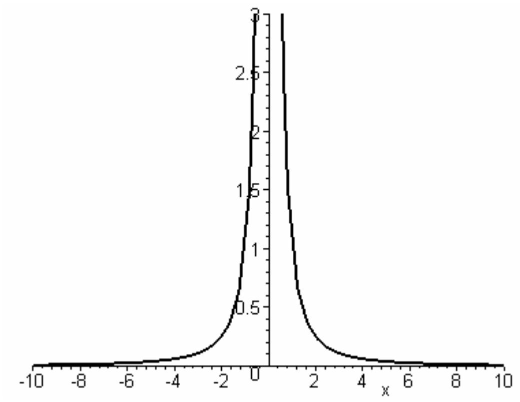


We observe that the graph of f(x) is not continuous at x = 1. However the graph of g(x) = x is continuous at each point in its domain.

The function is defined at x = 1 and equal 3,

$\lim_{x\to 1} f(x)=\lim_{x\to 1} x = 1$.

Example (2) The graph of the function $f(x)=\dfrac{1}{x^2}$ is not continuous at x = 0 (see below)



The function is not defined at x = 0,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{x^2} = \infty..$$

Now, we have the following definition of a continuous function:

Definition 1:

A function f(x) is continuous at a if and only if the following conditions are follows:

1- f(x) is defined at x = a, i.e., a is in the domain of f(x),

2- $\lim_{x\to a} f(x)$ exists,

3- $\lim_{x\to a} f(x) = f(a)$.

If f(x) doesn't satisfy one of the above conditions, it is said to be discontinuous at a.

Example (3) use the definition of continuity to show that the given function $f(x) = x^3 - 5x$ is continuous at x = 2.

Solution

1- $f(2) = (2)^3 - 5(2) = 8 - 10 = -2$,

2- $\lim_{x\to 2} f(x) = \lim_{x\to 2}(x^3 - 5x) = (2)^3 - 5(2) = 8 - 10 = -2$,

3- $\lim_{x\to 2} f(x) = f(2)$.

Now, all conditions are satisfied, i.e., f(x) is continuous at x = 2.

Example (4) use the definition of continuity to show that the function $f(x) = x^2 + \sqrt{7-x}$ is continuous at a = 4.

Solution

f(a) = f(4) = $4^2 + \sqrt{7-4} = 16 + \sqrt{3}$,

$$\lim_{x\to 4} f(x) = \lim_{x\to 4}(x^2 + \sqrt{7-x}) = \lim_{x\to 4} x^2 + \lim_{x\to 4}\sqrt{7-x} = 4^2 - \sqrt{\lim_{x\to 4}(7-x)} = 4^2 + \sqrt{7-4} = 16 + \sqrt{3} = f(4)$$

$\therefore \lim_{x\to 4} f(x) = f(4)$, then f(x) is continuous at x = 4.

Example (5) explain why the following function:

$$f(x)=\begin{cases}\dfrac{x^2-x}{x^2-1} & if \quad x\neq 1,\\ 1 & if \quad x=1,\end{cases}$$

is discontinuous at a = 1.

Solution

f(1) = 1,

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\left(\frac{x^2-x}{x^2-1}\right)=\lim_{x\to 1}\frac{x(x-1)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{x}{(x+1)}=\frac{1}{1+1}=\frac{1}{2}\neq f(1)\ \therefore\ \lim_{x\to 4} f(x)=f(4),$$

$\because \lim_{x\to 1} f(x)\neq f(1)$ then f(x) is not continuous at x = 1. The point x = 1, is then called **discontinuity** point.

Definition 2 A function f(x) is continuous from the right at a number (a) if

$$\lim_{x\to a^+} f(x)=f(a^+),$$

and f(x) is continuous from the left at a number (a) if

$$\lim_{x\to a^-} f(x)=f(a^-).$$

Definition 3 (Continuity of the function on an interval)

(1) If the function f is defined on the interval $I=(a,b)$, then f(x) is said to be continuous on I if f(x) is continuous at every point belongs to I.

(2) If the function is defined on the closed interval $I=[a,b]$, then f(x) is continuous on I if:

- (i) f(x) is continuous on the open interval (a,b),
- (ii) f(x) is continuous from the right to (a), i.e., $\lim_{x\to a^+} f(x)=f(a^+)$,
- (iii) f(x) is continuous from the left to (b), i.e., $\lim_{x\to b^-} f(x)=f(b^-)$

Example (6) show that the function $f(x)=x^2+\sqrt{8-x}$ is continuous on $[-1,1]$

Solution: let $-1<a<1$

$f(a)=a^2+\sqrt{8-a}$,

$$\lim_{x\to a} f(x)=\lim_{x\to 4}(x^2+\sqrt{8-x})=\lim_{x\to a} x^2+\lim_{x\to a}\sqrt{8-x}=4^2-\sqrt{\lim_{x\to a}(7-x)}=a^2+\sqrt{8-a}=f(a),$$

$\therefore \lim\limits_{x\to a} f(x) = f(a)$, this implies that function is continuous on the open interval $(-1,1)$, $\longrightarrow$ (i)

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (x^2 + \sqrt{8-x}) = (-1)^2 + \sqrt{8-(-1)} = 1 + \sqrt{9} = 1 + 3 = 4 = f(-1)$, $\longrightarrow$ (ii)

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x^2 + \sqrt{8-x}) = (1)^2 + \sqrt{8-(1)} = 1 + \sqrt{7} = f(1),$, $\longrightarrow$ (iii)

From (i), (ii) and (iii) and using definition 3, it is now clear that the function $f(x) = x^2 + \sqrt{8-x}$ is continuous on $[-1,1]$.

Theorem 1 If f(x) and g(x) are continuous at (a) and (c) is a constant, then the following functions are also continuous at (a):

(i) $f(x) \pm g(x),$

(ii) $c.f(x) \quad for \quad each \quad c \in R,$

(iii) $f(x).g(x)$,

(iv) $\dfrac{f(x)}{g(x)}$ except points that makes $g(x) = 0$.

Proof: [refer to the book, using the limits laws].

Important results:

(1) Any polynomial function is continuous on R,

(2) Any rational function is continuous on R except its zeros of the denominator,

Example (7) use the definition of continuity to show that the given function $f(x) = x^3 - 5x$ is continuous at x = 2.

Solution

1- $f(2) = (2)^3 - 5(2) = 8 - 10 = -2$,

2- $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}(x^3 - 5x) = (2)^3 - 5(2) = 8 - 10 = -2$,

3- $\lim\limits_{x\to 2} f(x) = f(2).$

Now, all conditions are satisfied, i.e., f(x) is continuous at x = 2.

Example (8) use the definition of continuity to show that the function $f(x)=\dfrac{x-2}{4x}$ is continuous at x = −2.

Solution

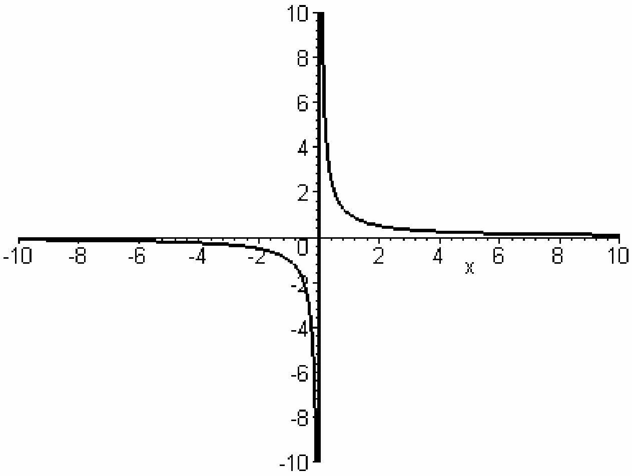
1- $f(-2)=\dfrac{(-2)-2}{4(-2)}=\dfrac{-4}{-8}=\dfrac{1}{2}$,

2- $\lim\limits_{x\to -2} f(x)=\lim\limits_{x\to -2}\dfrac{x-2}{4x}=\dfrac{-2-2}{4(-2)}=\dfrac{-4}{-8}=\dfrac{1}{2}$,

3- $\lim\limits_{x\to -2} f(x)=f(-2)$.

Now, all conditions are satisfied, i.e., f(x) is continuous at x = −2.

Example (9) the function $f(x)=\dfrac{1}{x}$ has a zero denominator when we put x = 0, so this function has a discontinuity at x = 0 (i.e., $f(x)=\dfrac{1}{x}$ is not continues at x = 0) and continuous at all other points. This fact can also be shown from the graph of f(x):



Example (10) Find all points of discontinuity of the following function:

$$f(x)=\frac{x^2+6x+9}{x^2+2x-15}.$$

Solution

First we calculate the zeros of the denominator by putting, $x^2+2x-15=0$, This implies that (x – 3)(x + 5) = 0. Then zeros of the denominator are $3 \ \ and \ -5$. Thus, f(x) is discontinuous at the points $3 \ \ and \ -5$ and continuous otherwise.

Example (11) Find all points of discontinuity of the following function:

$$f(x) = \frac{x-4}{x^2+1}.$$

Solution

First we calculate the zeros of the denominator by putting, $x^2 + 1 = 0,$ this implies that $x^2 = -1$ So, no zeros in the real numbers, therefore there is <u>no</u> points of discontinuity of this function.

(3) Trigonometric functions:
F(x) = sin (x) is continuous on R,
F(x) = cos (x) is continuous on R,
F(x) = tan (x) is continuous on $R - \{(2n+1)\frac{\pi}{2}, \quad n \in Z\}$,

(4) the inverse trigonometric functions are continuous at every number in their domain
for instance, $f(x) = \tan^{-1}(x)$ is continuous on R,

Example (12) show that the function $h(x) = \frac{\sin x}{x+1}$ is continuous at every point belongs to R, and state its domain.

Solution:

Since, F(x) = sin (x) is continuous on R, and g(x) = x + 1 is a polynomial also continuous on R, then using theorem (1) their quotient $\frac{f(x)}{g(x)} = h(x)$ is also continuous on $R - \{-1\}$.

(5) Radical (root) function continuous at every number in its domain.

(6) The logarithmic function is continuous at $(0, \infty)$,
The function $f(x) = \ln x$ is also continuous at $(0, \infty)$.

Theorem 2 If f(x) is continuous at b and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f[g(x)] = f(b)$. In other words we write:

$$\lim_{x\to a} f[g(x)] = f[\lim_{x\to a} g(x)].$$

For instance, $\lim_{x\to 0} \sin(x^2) = \sin \lim_{x\to 0} x^2 = \sin 0 = 0.$

Theorem 3 If g(x) is continuous at (a) and f(x) is continuous at g(a), then $(f \circ g)(x) = f[g(x)]$ is also continuous at (a).

Example (13) show where the function $h(x) = \sin(x^2)$ is continuous?.

Solution

Let $g(x) = x^2$ and $f(x) = \sin x$,

We know that $g(x) = x^2$ is continuous on R (since it is a type of a polynomial), and also the trigonometric function $f(x) = \sin x$ is continuous on R (as we have said before).

The composition of the two function f and g are $(f \circ g)(x) = f[g(x)] = h(x)$ has the domain R.

Now, using the theorem 3, the composition function $(f \circ g)(x) = \sin(x^2)$ is also continuous on R.

Theorem 4 (The intermediate value theorem):

Suppose that the function f is continuous on the closed interval $[a, b]$, and let N be a number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there exists a number c in the open interval (a, b) such that $f(c) = N$.

Example (14) use the intermediate value theorem (theorem 4) to prove that there is a positive number c such that $c^2 = 2$.

Solution

The function $f(x) = x^2$ is continuous on the closed interval $[a, b] = [1, 2]$.

$f(a) = f(1) = (1)^2 = 1,$

$f(b) = f(2) = (2)^2 = 4,$

It is clear that the number 2 lies between f(a) and f(b), Let $N = 2$, then using theorem 4, there exists a number c in the open interval $(1, 2)$ such that $f(c) = 2$.

But $f(c)=c^2$. Thus, there exists a number c in the open interval $(1,\ 2)$ such that $c^2=2$.

Remark: the last example proves the existence of $\sqrt{2}$ in the interval $(1,\ 2)$.

Example (15) prove that the equation $\sqrt{x-5}=\dfrac{1}{x+3}$ has at least one real root.

Solution for the equation $\sqrt{x-5}=\dfrac{1}{x+3}$, by squaring the two sides we obtain

$(x-5)=\dfrac{1}{(x+3)^2}$, this implies that $(x-5)(x+3)^2=1$, then $(x-5)(x+3)^2-1=0$.

Now, let $f(x)=(x-5)(x+3)^2-1$, since f(x) is a polynomial function, then f(x) is continuous on R. This implies that f(x) is continuous on any interval of R, for instance f(x) is continuous on the closed interval [5, 6]

$f(5)=0-1=-1<0,$

$f(6)=(1)(9)^2-1=81-1=80>0$.

Now the number 0 lies between f(5) and f(6), Let $N=0$, then using theorem 4, there exists a number c in the open interval $(5,\ 6)$ such that $f(c)=0$. i.e, there exists at least one root c of the polynomial f(x) in the open interval $(5,\ 6)$.

Thus, the equation $\sqrt{x-5}=\dfrac{1}{x+3}$ has at least one root in the interval $(5,\ 6)$.

Home work: solve pages 133 and 134 in your book, problems No. 3, 7, 11, 12, 15, 16, 19, 29, 34, 39, 42, 51, 52, 59 and 60.