Partial review:

The function:

The function is a relation between two sets A and B, such that each element x in A has a unique image y in B, we then say that f(x) = y.

In the function f(x) = y, the letter x is called independent variable, and y is called dependent variable.

The set of all independent variables-A here-is called the domain of the function, and the set of all dependent variables-B here-is called the range of the function.

Types of functions:

- The polynomial function:

  The general form of the polynomial function of degree n is

  $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$

  When n = 1, the polynomial function becomes linear function and when n = 2, it becomes the quadratic function and so on.

  The domain of the polynomial function of any degree is the set of all real numbers R.

- The rational function:

  The rational function is always written as a quotient of two function, for instance

  $f(x) = \frac{g(x)}{h(x)}$, the domain of the rational function is set of all real numbers except zeros of denominator. i.e, domain = $R - \{zeros\ of\ \ h(x)\}$.

- The radical function:

  The radical function always equal expression under the radical sign, for instance

  $f(x) = \sqrt{E(x)}, \quad where\ \ E(x)\ is\ a\ mathematical\ \ \exp ression.$ To find its domain, put the expression under the radical ≥ 0 , then the domain will be $\{ x | \ x \in R,\ and\ \ E(x) \geq 0\}$.

- The logarithmic function:

  For instance, $f(x) = \log_a^x$, and $f(x) = \ln_e^x$, … .

  The number $e \approx 2.71828$ is always the base of the ln function, therefore we briefly write f(x) = ln(x).

- Trigonometric function:

  For instance,

  $$f(x)=\sin(x),\quad f(x)=\cos(x),\quad f(x)=\tan(x)=\frac{\sin(x)}{\cos(x)},\quad f(x)=\sec(x)=\frac{1}{\cos x},\ f(x)=\csc(x)=\frac{1}{\sin x}$$

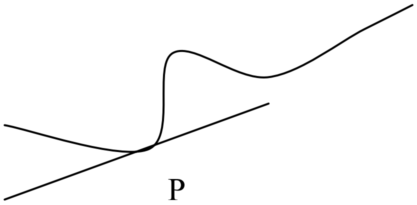
- Case defined function:

  For instance the modulus function $f(x)=|x|$, which is defined as follows,

  $$|x|=\begin{cases}x, & \text{if} \quad x>0\\ -x & \text{if} \quad x<0.\end{cases}$$

The tangent and velocity problems:

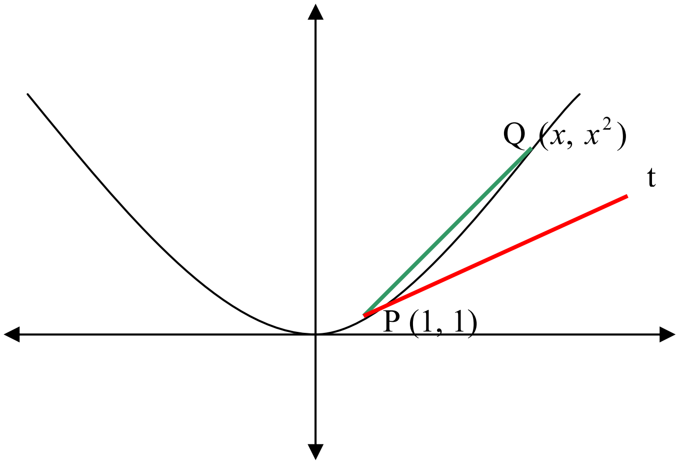
A tangent to a curve is a line that touches the curve at a point.



Example (1) Find an equation of the tangent line to the parabola $y=x^2$ at the point (1,1).

Solution:

We choose a point Q $(x,\ x^2)$ close to the point P (1,1), and the line t (red line) is the tangent line to the parabola at the point P. The line segment PQ is secant line to the parabola (see the graph).



The slope of the line segment PQ is $m_{\overline{PQ}}$, and the slope of the tangent line t is m.

Since slope $= \dfrac{\textit{difference in y coordinates}}{\textit{difference in x coordinates}}$, then the slope of the secant line PQ is given as:

$$m_{\overline{PQ}} = \frac{x^2-1}{x-1}.$$

Notice that we can't put x = 1, to avoid division by zero. Imagine the point Q is moving to approach the point P (but not equal to P). The slopes of the corresponding secant lines is then given by the relation

$$m_{\overline{PQ}} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1,$$

And when Q is very close to P this means that the x coordinates of Q is very close to 1 (but not equal), in the following table we have calculated some values of x near 1.

x	0.9	0.99	0.999	$\xrightarrow{\cdots\cdots}$	1	$\xleftarrow{\cdots\cdots}$	1.01	1.5	2
$m_{\overline{PQ}}$	1.9	1.99	1.999	$\xrightarrow{\cdots\cdots}$	2	$\xleftarrow{\cdots\cdots}$	2.01	2.5	3

From the table we observe that the value of the slopes $m_{\overline{PQ}}$ approaches to the value 2 to as x approaches to the value 1, i.e $m_{\overline{PQ}} \to 2$ as $x \to 1$, and this is read as: $m_{\overline{PQ}}$ approaches 2 as x approaches 1. This arrows notation can be written in another form as:

$$\underset{x\to 1}{Lim}\ m_{\overline{PQ}} = 2.$$

When the point Q approaches the point P then the secant line (green line) will coincide on the tangent line t (the red one) the slope of the tangent line t . Thus, the slope of the tangent line t is the limit of the slopes of the secant lines. Therefore, $\underset{x\to 1}{Lim}\ m_{\overline{PQ}} = m.$ Now, the slope of the tangent line t is given as $m \approx 2.$

Now, to find the equation of the tangent line t passing through the point P (1,1) at the curve of the parabola, we have just calculated its slope $m \approx 2.$ and given from the equation:

$$\frac{y-1}{x-1} = 2,$$

This implies that the equation of the tangent line t is $y = 2x - 1.$

Now, consider the rational function $f(x) = m_{\overline{PQ}} = \dfrac{x^2-1}{x-1}$, we can see that 1 doesn't belong to the domain of this function and from the previous example

$$\underset{x\to 1}{Lim}\, f(x) = 2.$$

We then say that "***the limit of the function f(x) equal 2 as x approaches to 1***".

In general, we say that $\underset{x\to a}{Lim}\, f(x) = L,$ when f(x) approaches L as x approaches a.

In example (1) we noticed when $x < 1 \;\Rightarrow\; f(x) \to 2$ (see the table), this is called the Left-hand limit of f(x), and is written as $\underset{x\to 1^-}{Lim}\, f(x) = 2.$

And also when $x > 1 \Rightarrow\; f(x) \to 2$ (see the table), this is called the Right-hand limit of f(x), and is written as $\underset{x\to 1^+}{Lim}\, f(x) = 2.$

Theory (existence of the limit): ***The limit of a function exists if and only if the Left-hand limit equals the Right-hand limit of this function.***

In our example the left hand limit equal the right hand limit $\underset{x\to 1^-}{Lim}\, f(x) = \underset{x\to 1^+}{Lim}\, f(x) = 2.$

Then based on the above rule the limit of $f(x) = \dfrac{x^2-1}{x-1}$ exists.

Example (2) determine the left hand and right hand limits (as x approaches 2) of the following function:

$$f(x) = \begin{cases} x+1, & x > 2, \\ x-1, & x < 2. \end{cases}$$

Solution:

$\underset{x\to 2^-}{Lim}\, f(x) = \underset{x\to 2^-}{Lim}(x-1) = (2-1) = 1.$ i.e., the left hand limit of this function equals 1.

$\underset{x\to 2^+}{Lim}\, f(x) = \underset{x\to 2^+}{Lim}(x+1) = (2+1) = 3.$ i.e., the left hand limit of this function equals 3.

Since $\underset{x\to 2^-}{Lim}\, f(x) \neq \underset{x\to 2^+}{Lim}\, f(x)$, then $\underset{x\to 2}{Lim}\, f(x)$ doesn't exist.

Example (3) find $\underset{x\to 0}{Lim}\dfrac{|x|}{x}$ (if it exists).

Solution:

$$\frac{|x|}{x}=\begin{cases}\frac{x}{x}=1, & x>0,\\ \frac{-x}{x}=-1, & x<0.\end{cases}$$

$\underset{x\to 0^-}{Lim}\frac{|x|}{x}=\underset{x\to 0^-}{Lim}(-1)=-1.$ i.e., the left hand limit of this function equals -1.

$\underset{x\to 0^+}{Lim}\frac{|x|}{x}=\underset{x\to 0^+}{Lim}(+1)=1.$ i.e., the left hand limit of this function equals 1.

Since $\underset{x\to 0^-}{Lim}\frac{|x|}{x}\neq\underset{x\to 0^+}{Lim}\frac{|x|}{x}$, then $\underset{x\to 0}{Lim}\frac{|x|}{x}$ doesn't exist.

Infinite limits:

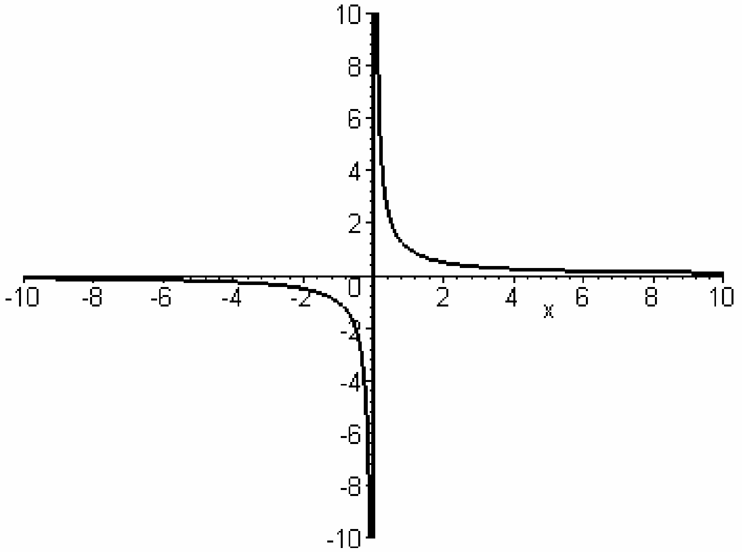
This means that the value of the limit of the function approaches to ∞ or $-\infty$.

Example (4) Find the following limits (if it exists)

$(i)\ \lim_{x\to 0}\frac{1}{x}$, $(ii)\ \lim_{x\to 0}\frac{1}{x^2}$, $(iii)\ \lim_{x\to 0^+}\frac{-3}{x}$

Solution

(i) by graphing the equation $f(x)=\frac{1}{x}$ we have

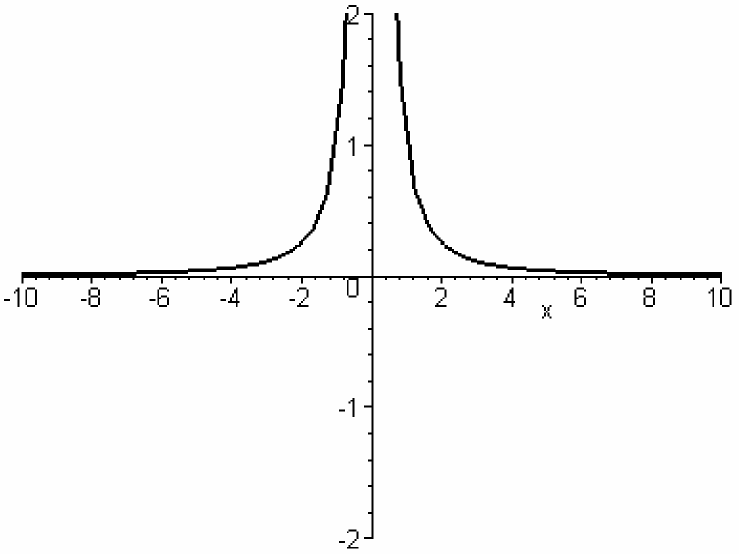


Thus, $\lim_{x\to 0^+}\frac{1}{x}=\infty$, and $\lim_{x\to 0^-}\frac{1}{x}=-\infty$,

$\because \quad \lim_{x\to 0^+} \frac{1}{x} \neq \lim_{x\to 0^-} \frac{1}{x}$

$\therefore \lim_{x\to 0} \frac{1}{x}$ doesn't exist.

(ii) by graphing the equation $f(x) = \frac{1}{x^2}$ we have

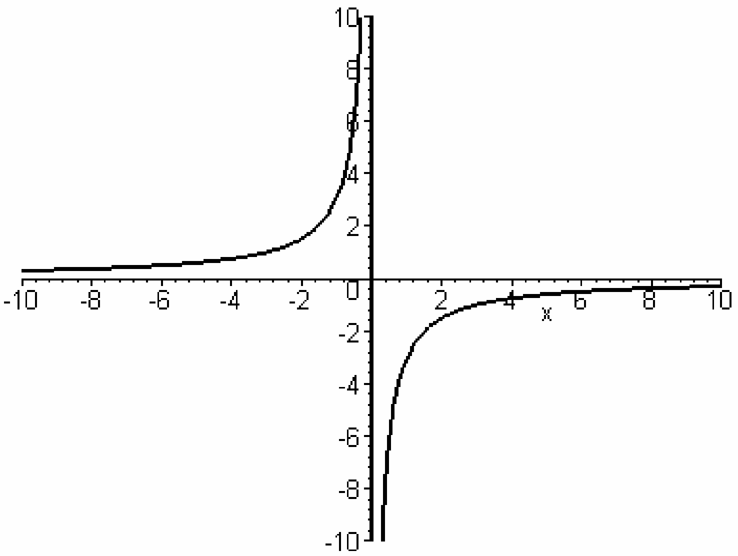


Thus, $\lim_{x\to 0^+} \frac{1}{x^2} = \infty,$ and $\lim_{x\to 0^-} \frac{1}{x^2} = \infty,$

$\because \quad \lim_{x\to 0^+} \frac{1}{x} = \lim_{x\to 0^-} \frac{1}{x} = \infty$

$\therefore \lim_{x\to 0} \frac{1}{x^2}$ exists.

(iii) by graphing the equation $f(x) = -\frac{3}{x}$ we have



Thus, $\lim_{x\to 0^+} \frac{-3}{x} = -\infty$

The limit also doesn't exist at x = 0.

Example (5) Find $\lim_{x\to 1} \frac{1}{x-1}$ $(if \ \ it \ \ exists)$

Solution

$\lim_{x\to 1^+} \frac{1}{x-1} = \infty$, and $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$.

$\because \quad \lim_{x\to 1^+} \frac{1}{x-1} \neq \lim_{x\to 1^-} \frac{1}{x-1}$

$\therefore \lim_{x\to 1} \frac{1}{x-1}$ doesn't exist.

Example (6) Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

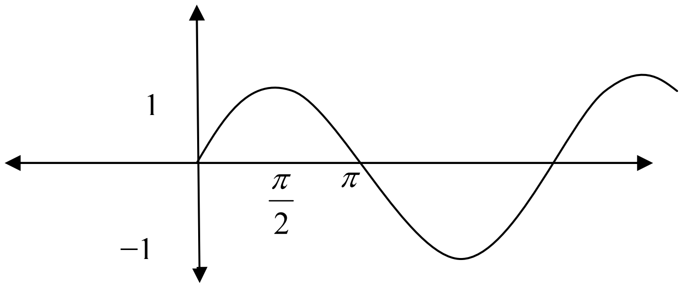
Solution:

$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$ and $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$. Then $\lim_{x\to 3} \frac{2x}{x-3}$ doesn't exist.

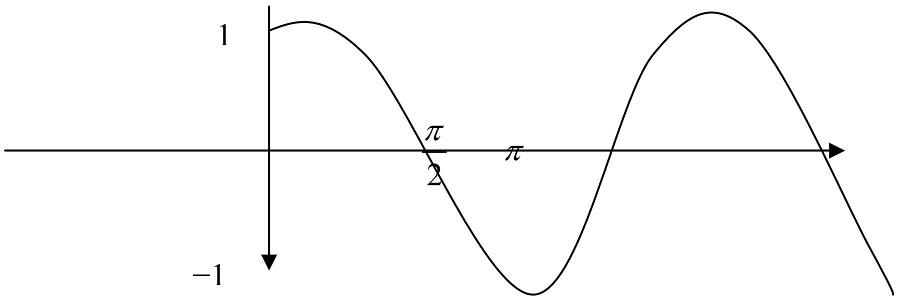
Example (7) find $\lim_{x\to \frac{\pi}{2}} \tan x$ $(if \ \ it \ \ exists)$

Solution : since $Tan \ x = \frac{\sin x}{\cos x}$, then by graphing sin x and cos x we can find this limit

The plot of the function f(x) = sin(x)



The plot of the function f(x) = cos(x)



Now, $\lim_{x\to\frac{\pi}{2}} \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\sin(x)}{\cos(x)}$,

Since, as $x \to \frac{\pi}{2}^{+} \Rightarrow \cos(x) \to 0^{-}$ and sin (x) is always positive as $x \to \frac{\pi}{2}^{+}$ and $x \to \frac{\pi}{2}^{-}$ (see the graphs of sin and cos functions). Then:

$$\lim_{x\to\frac{\pi}{2}^{+}} \tan x = \lim_{x\to\frac{\pi}{2}^{+}} \frac{\sin(x)}{\cos(x)} = -\infty, \text{ and } \lim_{x\to\frac{\pi}{2}^{-}} \tan x = \lim_{x\to\frac{\pi}{2}^{-}} \frac{\sin(x)}{\cos(x)} = \infty.$$

Another way of finding this limit by graphing the tan function we obtain the same results.

Example (8) determine $\lim_{x\to\pi} \csc x$ (*if it exists*)

Solution : since $\csc x = \frac{1}{\sin x}$, then by graphing sin x, we can find this limit

Now, $\lim_{x\to\pi} \csc x = \lim_{x\to\pi} \frac{1}{\sin(x)}$, Since, as $x \to \pi^{+} \Rightarrow \sin(x) \to 0^{-}$ and as $x \to \pi^{-} \Rightarrow \sin(x) \to 0^{+}$ (see the graphs of sin function).

Then:

$$\lim_{x\to\pi^{+}} \csc x = \lim_{x\to\pi^{+}} \frac{1}{\sin(x)} = -\infty, \text{ and } \lim_{x\to\pi^{-}} \csc x = \lim_{x\to\pi^{-}} \frac{1}{\sin(x)} = \infty.$$

Example (9) discuss the existence of $\lim_{x\to 0} \sin\frac{1}{x}$

Solution

As $x \to 0$, this implies that $\frac{1}{x} \to \infty$. Thus $\lim_{x\to 0} \sin\frac{1}{x}$ is equivalent to find the value of the angle x as x becomes a very big number, i.e., sin(∞).

However, from the plot of the sin function $f(y) = \sin y$, we can see the function is alternating between the two values 1 and −1 as and doesn't have a definite value (because it is periodic).

Thus, $\lim_{x\to 0} \sin\frac{1}{x}$ doesn't exist.

Home work: Solve pages (102) and (103) in your book, problems No. 5, 7, 23, 24, 26, 27, 32.