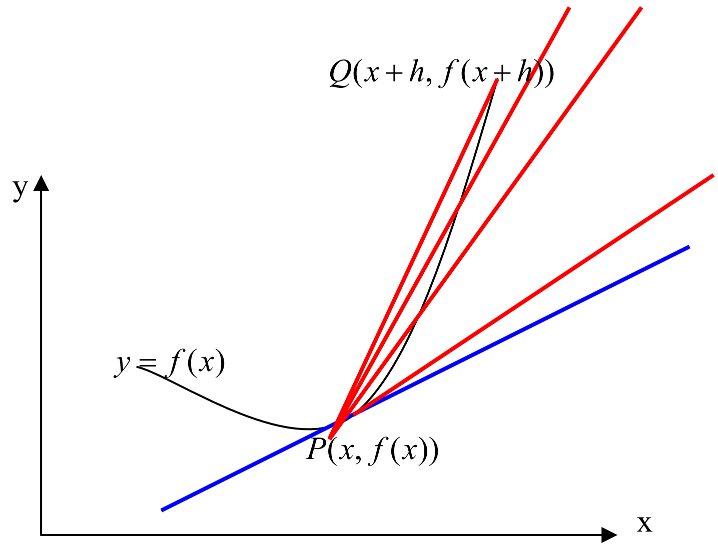
The derivatives



Consider the point P on the curve of the function f(x) with coordinate (x,y) and the point Q is sufficiently small increasement of the point P with coordinates $(x+h, y+h)$.

The slope of the secant line PQ is m_{PQ} and given by:

$$m_{PQ} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

When the point Q moves on the curve of f(x) to reach P, the secant lines (red lines) will move to coincide with the tangent line-blue line- of the point P (this only happens as h approaches to zero).

Thus, the limiting behavior of the slope of the secant lines approaches to the slope of the curve at the point b as h tends to zero. We summarize our results by writing:

$$\lim_{h\to 0} m_{PQ} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = m.$$

where m is the slope of the tangent line to the curve of f(x) at the point P.

The function which generates all the slopes of the tangent lines to the curve of y = f(x) at a point (x,y) is called the derivative of f(x). Now, we have to introduce the following definition:

Definition 1

The derivative of a function f(x) is the function denoted f' (read "f prime") and defined by

$$f' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that this limit exists.

Also, f' is the limit of the difference quotient as $h \to 0$.

Remark, the following symbols of writing the derivative of the function f(x) are equivalent:

$$f'(x) = y' = \frac{df(x)}{dx} = \frac{dy}{dx}.$$

Remarks:

- If $f'(x)$ exists then f(x) is said to <u>differentiable</u> at x.
- The derivative of a function f(x) at a point x_1 is the slope of the tangent line at the point x_1. i.e., $f'(x_1)$ is the slope of the line tangent to the graph y = f(x) at the point $(x_1, f(x_1))$, other writing:

$$\left.\frac{dy}{dx}\right|_{x=x_1} = y'(x_1).$$

Example 1 use the definition of the derivative to find $f'(x)$ where $f(x) = x^2$.

Solution:

Since $f(x) = x^2$, then $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$$\therefore \frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + h^2}{h} = 2x + h,$$

$$\therefore f' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x + 0 = 2x.$$

Example 2 use the definition of the derivative to find $\frac{d}{dx}(\sqrt{x})$.

Solution:

Since $f(x) = \sqrt{x}$, then $f(x+h) = \sqrt{x+h}$

$$\therefore \frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h},$$

$$\therefore f'=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}.\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{(x+h)-x}{h.(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h.(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x})}=\frac{1}{(\sqrt{x+0}+\sqrt{x})}=\frac{1}{2\sqrt{x}}.$$

Example 3 find $\frac{dp}{dq}$ if $p=q^2+3q-1$

***Solution*:**

Since $p=q^2+3q-1$ then $p(q+h)=(q+h)^2+3(q+h)-1=q^2+2qh+h^2+3q+3h-1$

$$\therefore \frac{f(x+h)-f(x)}{h}=\frac{(q^2+2qh+h^2+3q+3h-1)-(q^2+3q-1)}{h}$$

$$=\frac{q^2+2qh+h^2+3q+3h-1-q^2-3q+1}{h}=2q+h+3.$$

$$\therefore f'=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(2q+h+3)=2q+0+3=2q+3.$$

Example 4 find $\frac{d}{dx}(x+5)^2$ using the definition

***Solution*:**

Since $f(x)=(x+5)^2$, now we need to calculate $\frac{df(x)}{dx}=f'(x)$

$f(x+h)=(x+5+h)^2=x^2+h^2+25+2xh+10x+10h$

$$\therefore \frac{f(x+h)-f(x)}{h}=\frac{x^2+h^2+25+2xh+10x+10h-(x^2+10x+25)}{h}$$

$$=\frac{x^2+h^2+25+2xh+10x+10h-x^2-10x-25}{h}=h+2x+10.$$

$$\therefore f'=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(h+2x+10)=0+2x+10=2x+10=2(x+5).$$

Example (5) in the last example find the equation of the tangent line to the curve of $y=(x+5)^2$, at the point $(1,36)$.

Solution $\left.\frac{dy}{dx}\right|_{x=x_1} = y'(x_1) = m$ is the slope of the tangent to the curve y = f(x) at the point x_1, where $x_1 = 1$.

Since $f'(x) = 2(x+5)$

$\therefore \left.\frac{dy}{dx}\right|_{x=x_1} = y'(x_1) = 2(x_1 + 5) = 2(1+5) = 12.$

Thus, the line tangent to the curve of the function $f(x) = (x+5)^2$ at the point $(1,36)$ has the slope m = 12. To find the equation of this line , we know that the line passes through the point $(1,36)$.

Using the formula $slope = m = \frac{y - y_1}{x - x_1}$, where $(x_1,\ y_1) = (1,36)$,

$\therefore 12 = \frac{y-36}{x-1}$,

$\therefore y - 36 = 12(x-1), \Rightarrow y = 12x - 12 + 36,$

Thus, the required equation is $y = 12x + 24$.

Home work: solve the book pages 544 and 545, the following problems:

Use the definition of the derivative to find each of the following:

[13] $\frac{dp}{dq}$ if $p = 2q^2 + 5q - 1,$

[14] $\frac{d}{dx}(x^2 - x - 3)$,

[17] $f'(x)$ if $f(x) = \sqrt{x+2}$,

[27] find an equation of the tangent line to the curve $y = (x-7)^2$ at the point (6,1),

[27] find an equation of the tangent line to the curve $y = \frac{3}{x-1}$ at the point (2,3).