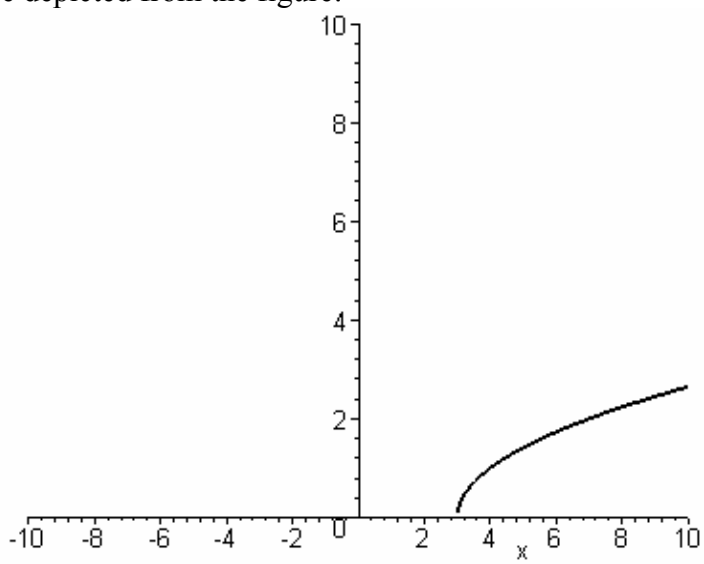
Example (7) Find $\lim_{x\to 3^+}\sqrt{x-3}$

Solution

The domain of the function is defined for $x \geq 3$. Thus,

$$\lim_{x\to 3^+}\sqrt{x-3} = \lim_{x\to 3^+}\sqrt{3-3} = \lim_{x\to 3^+}\sqrt{0} = 0.$$

This result can also be depicted from the figure:



Example (8) $\lim_{x\to 1^+} 4\sqrt{x-1}$

Solution

The domain of the function is defined for $x \geq 3$. Thus,

$$\lim_{x\to 1^+} 4\sqrt{x-1} = \lim_{x\to 1^+} 4\sqrt{1-1} = \lim_{x\to 1^+} 4\sqrt{0} = 0.$$

Example (9) Let

$$f(x) = \begin{cases} x & if \quad x \leq 1 \\ 2 & if \quad x > 1 \end{cases}$$

Find (i) $\lim_{x\to 1^+} f(x)$, (ii) $\lim_{x\to 1^-} f(x)$, (iii) $\lim_{x\to 1} f(x)$.

Solution

(i) $\lim_{x\to 1^+} f(x) = 2$,

(ii) $\lim_{x\to 1^-} f(x) = 1$,

(iii) since $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$ then $\lim_{x\to 1} f(x)$ doesn't exist.

Infinite limits

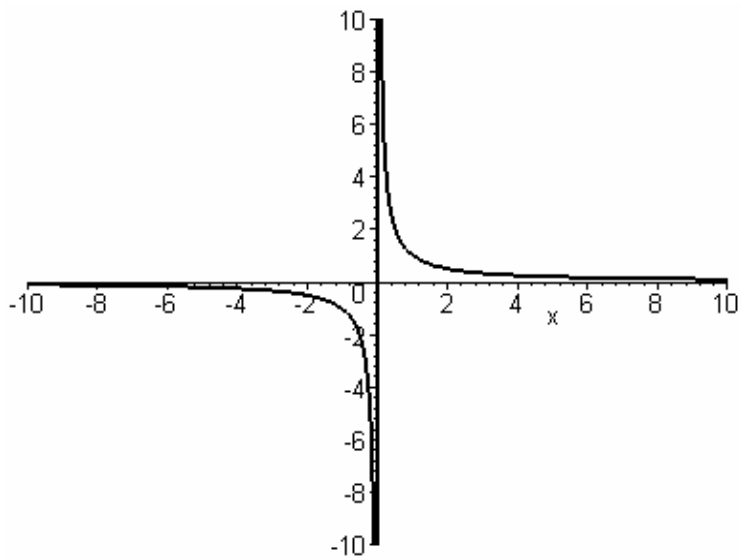
Now, we will examine the limits of the form $\frac{A}{0}$ (where a is a real number, $A \neq 0$).

Example (10) Find the following limits (if it exists)

$(i)\ \lim_{x\to 0}\frac{1}{x}$, $(ii)\ \lim_{x\to 0}\frac{1}{x^2}$, $(iii)\ \lim_{x\to 0^+}\frac{-3}{x}$

Solution

(i) by graphing the equation $f(x)=\frac{1}{x}$ we have

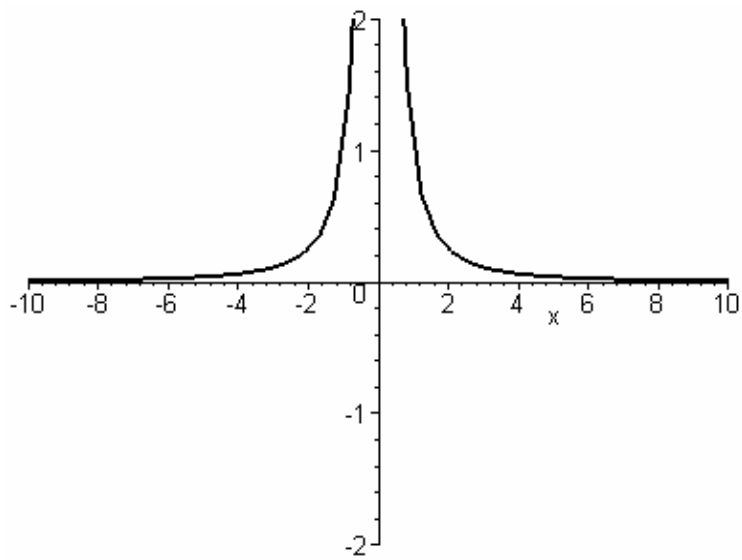


Thus, $\lim_{x\to 0^+}\frac{1}{x}=\infty$, and $\lim_{x\to 0^-}\frac{1}{x}=-\infty$,

$\because \quad \lim_{x\to 0^+}\frac{1}{x} \neq \lim_{x\to 0^-}\frac{1}{x}$

$\therefore\ \lim_{x\to 0}\frac{1}{x}$ doesn't exist.

(ii) by graphing the equation $f(x)=\frac{1}{x^2}$ we have

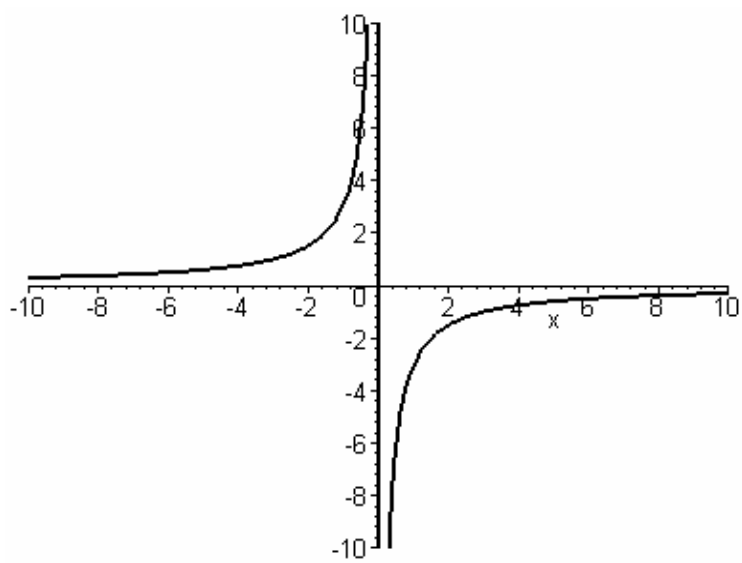


Thus, $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = \infty$, and $\lim\limits_{x\to 0^-} \dfrac{1}{x^2} = \infty$,

$\because \quad \lim\limits_{x\to 0^+} \dfrac{1}{x} = \lim\limits_{x\to 0^-} \dfrac{1}{x} = \infty$

$\therefore \lim\limits_{x\to 0} \dfrac{1}{x^2}$ exists.

(iii) by graphing the equation $f(x) = -\dfrac{3}{x}$ we have



Thus, $\lim\limits_{x\to 0^+} \dfrac{-3}{x} = -\infty$

Example (11) Find $\lim_{x\to 1}\frac{1}{x-1}$ $(\textit{if it exists})$

Solution

$$\lim_{x\to 1^+}\frac{1}{x-1}=\infty, \quad \text{and} \quad \lim_{x\to 1^-}\frac{1}{x-1}=-\infty.$$

$$\because \quad \lim_{x\to 1^+}\frac{1}{x-1}\neq \lim_{x\to 1^-}\frac{1}{x-1}$$

$\therefore \lim_{x\to 1}\frac{1}{x-1}$ doesn't exist.

Limits at infinity

$$\lim_{x\to\infty}\frac{1}{x}=0 \qquad \text{and} \qquad \lim_{x\to-\infty}\frac{1}{x}=0$$

In general,

$$\lim_{x\to\infty}(\frac{1}{x})^p=0 \qquad \text{and} \qquad \lim_{x\to-\infty}(\frac{1}{x})^p=0 \quad \text{where} \quad p>0.$$

Example (12) Find $\lim_{x\to\infty}\frac{2x^2+5}{3x^2+6x+4}$

Solution by dividing both numerator and denominator by x^2

$$\lim_{x\to\infty}\frac{2\frac{x^2}{x^2}+\frac{5}{x^2}}{3\frac{x^2}{x^2}+6\frac{x}{x^2}+\frac{4}{x^2}}=\frac{\lim_{x\to\infty}2+5\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}3+6\lim_{x\to\infty}\frac{1}{x}+4\lim_{x\to\infty}\frac{1}{x^2}}=\frac{2+0}{3+6(0)+4(0)}=\frac{2}{3}.$$

Example (13) Find $\lim_{x\to-\infty}\frac{x+1}{x}$

Solution

$$\lim_{x\to-\infty}\frac{x+1}{x}=\lim_{x\to-\infty}(1+\frac{1}{x})=1+\lim_{x\to-\infty}\frac{1}{x}=1+0=1.$$

Limits at infinity for polynomials

Example (14) Find $\lim_{x\to\infty}(5x^2-3x)$

Solution

At infinity the term $5x^2$ becomes such large that we can neglect the value of the other term $-3x$

Now, $\lim_{x\to\infty}(5x^2-3x)=\lim_{x\to\infty}5x^2=\infty$.

Example (15) Find $\lim_{x\to\infty}(x^3-x^2+x-2)$

Solution

At infinity the term x^3 becomes such large that we can neglect the value of the other terms $-x^2+x-2$

Now, $\lim_{x\to\infty}(x^3-x^2+x-2)=\lim_{x\to\infty}x^3=\infty$..

Example (16) Find $\lim_{x\to-\infty}(x^3-x^2+x-2)$

Solution

$\lim_{x\to-\infty}(x^3-x^2+x-2)=\lim_{x\to-\infty}x^3=-\infty.$

Example (17) If

$$f(x)=\begin{cases} x & if \quad x\le 1 \\ 3 & if \quad x>1 \end{cases}$$

Find (i) $\lim_{x\to\infty}f(x)$, (ii) $\lim_{x\to-\infty}f(x)$

Solution

(i) $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}3=3.$

(ii) $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}x=-\infty.$