Quantum Zakharov Equations

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We investigate the existence of soliton solutions for modified Zakharov equations describing modulational instabilities in quantum plasmas. In particular, we show that quantum effects break down the existence of easily identifiable soliton solutions in the adiabatic limit. Instead of the integrable nonlinear Schrödinger equation, we show that the adiabatic limit of the quantum Zakharov equation is a coupled fourth-order system, not amenable to straightforward integration. Considering the simultaneous adiabatic and formal semi-classical limits, we obtain more detailed results through a variational solution. Quantum effects enhance dispersion, smearing out the classical one-soliton solution.

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1. Introduction

There has been an increased interest on the relevance of quantum effects for plasmas under extreme conditions as in laser or some astrophysical plasmas, as well as in ultra-small electronic devices [1]-[5]. This has motivated the introduction of approximated models describing plasmas with quantum corrections. One of these proposals is the multistream quantum model presented in ref. [1] and generalized in [2] to incorporate statistical effects. Applied to a two-stream system, the multistream quantum plasma model predicts some new instabilities (attenuated by statistics [2]) as well as an overall stabilization for large wave numbers. The same dispersive, stabilizing contribution appears for quantum effects in three-stream plasmas [3]. Similar behavior occurs for quantum plasmas described within the framework of the Wigner-Poisson system [4, 5].

Nonlinear phenomena in classical plasmas are frequently formulated in terms of completely integrable evolution equations of Korteweg-de Vries (KdV) or nonlinear Schrödinger equation (NLS) type [6]-[8]. These completely integrable equations, as is well known, admit N-soliton solutions derivable from the Inverse Scattering Transform (IST) method. Soliton solutions follow from a delicate balance between nonlinearity and dispersion, and a natural question concerns the role quantum effects play in this regard. For instance, applying the same weak nonlinearity expansion used for classical plasmas, now for the quantum hydrodynamic model [9], one obtains a modified KdV equation [10] describing the evolution of ion-acoustic waves in plasmas. As shown in [10], quantum effects can distort or even suppress one-soliton solutions, as a result of quantum dispersion effects. Following this trend, recently the interaction between ion-acoustic and Langmuir waves in quantum plasmas has been studied [11], in the framework of Zakharov equations [6]-[8] with quantum corrections. The linear modes of this system [11] are shown to have stronger stability properties in comparison with the classical plasma case. Nonlinear waves for the quantum Zakharov equations, however, are not yet sufficiently well understood. For instance, the classical Zakharov equations in the adiabatic limit reduce to the NLS equation, completely integrable in terms of the IST method. No similar technique seems to be easily constructed in the quantum case.

The purpose of this work is the investigation of the role played by quantum effects in the nonlinearity-dispersion balance leading to the formation of solitons in the Zakharov equations. We proceed by using a variational formulation for the simultaneous adiabatic and semi-classical regimes of the quantum Zakharov equations.

2. Zakharov equations with a quantum correction

Consider a two-species plasma constituted by electrons and ions, the ions being much more massive than electrons. Two different time scales are distinguished, the slow time scale of the ions and the fast time scale of the electrons, we can proceed with a two-time scale analysis. The low mobility of the ions as compared to that of the electrons justifies this kind of treatment. Considering the quantum hydrodynamic model for plasmas with a quantum correction [9, 10] and following the same procedure [6]-[8] as in the classical plasma case, we obtain the following modified quantum Zakharov equations in one spatial variable,

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} - H^2\frac{\partial^4 E}{\partial x^4} = nE\,, \tag{2.1}$$

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} + H^2 \frac{\partial^4 n}{\partial x^4} = \frac{\partial^2 |E|^2}{\partial x^2}\,. \tag{2.2}$$

In the above quantum Zakharov equations, non-dimensional variables are used, E is an envelope electric field and n is the plasma density measured from its equilibrium value. In addition,

$$H = \frac{\hbar \omega_i}{\kappa_B T_e} \tag{2.3}$$

is a parameter measuring the importance of quantum effects in modulational instabilities, for ω_i the ion plasma frequency and T_e the temperature of the electron fluid. Notice that the quantum parameter H is build as the ratio of an ionic term, the ion plasmon energy, and an electronic term, the electron kinetic energy. In fact, the coupling between inertial (ionic) and dynamic (electronic) terms is characteristic of phenomena involving ion-acoustic wave propagation in plasmas, as in the case of the ion-acoustic wave velocity $c_s = (\kappa_B T_e/m_i)^{1/2}$, where m_i is the ion mass. In the formal classical limit $H \to 0$, the quantum Zakharov equations recover the classical [6]-[8] Zakharov system. For the details on the derivation of (2.1-2.2), we refer to [11].

The solutions of the classical ($H = 0$) Zakharov system, in the adiabatic limit, are found with $\partial^2 n/\partial t^2 \approx 0$ in (2.2). In this situation, the envelope electric field satisfies a nonlinear Schrödinger equation, which is completely integrable admitting N-soliton solutions. A natural question is if the quantum effects perturbs or even destroys the existence of these localized, solitonic solutions. Indeed, since solitons often arise as a balance between dispersive and nonlinear contributions, and since quantum effects enhance dispersion, we expect that quantum solitons are not so easily found for quantum Zakharov equations. Let us investigate this hypothesis taking $\partial^2 n/\partial t^2 \approx 0$ in (2.2). For suitable boundary conditions, this gives immediately

$$n = -|E|^2 + H^2 \frac{\partial^2 n}{\partial x^2}\,. \tag{2.4}$$

Equation (2.4), inserted in Eq. (2.1), yields

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + |E|^2 E = H^2 \left(\frac{\partial^4 E}{\partial x^4} + E\frac{\partial^2 n}{\partial x^2}\right)\,. \tag{2.5}$$

In the formal classical limit $H \to 0$, the right-hand side of Eq. (2.5) vanishes and we obtain the NLS equation, which is completely integrable. However, the traditional reduction procedure of searching for solutions in the form

$$E = F(x - Mt)\exp(i[k(x - ut) + \delta])\,, \quad n = G(x - Mt)\,, \tag{2.6}$$

for real functions F and G and real parameters k, M, u and δ does not seem to produce good results, since then a complicated fourth-order system of coupled, nonlinear equations arise. The existence of localized or soliton solutions for this system remains an open question.

Some insight in the question can be gained by considering the simultaneous adiabatic and semi-classical limits. Substituting (2.4) into (2.5) and retaining only up to $O(H^2)$ terms produce the decoupled equation

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} + |E|^2 E = H^2 \left(\frac{\partial^4 E}{\partial x^4} - E\frac{\partial^2 |E|^2}{\partial x^2}\right)\,. \tag{2.7}$$

Equation (2.7) is derivable from a variational principle,

$$\delta S = \delta \int \mathcal{L} dx dt = 0\,, \tag{2.8}$$

for the Lagrangian density

$$\mathcal{L} = \frac{i}{2}(E\frac{\partial E^*}{\partial t} - E^*\frac{\partial E}{\partial t}) + \frac{\partial E}{\partial x}\frac{\partial E^*}{\partial x} - \frac{|E|^4}{2} + H^2\frac{\partial^2 E}{\partial x^2}\frac{\partial^2 E^*}{\partial x^2} - \frac{H^2}{2}|E|^2\frac{\partial^2 |E|^2}{\partial x^2}\,. \tag{2.9}$$

As can be easily checked, the variational derivatives $\delta S/\delta E^* = \delta S/\delta E = 0$ produces (2.7) and its complex conjugate equation, respectively.

Inspired by the form of the classical soliton solutions, we propose time-dependent variational solutions of the class

$$E = \alpha(t)\exp(i\theta(t))\mathrm{sech}(\beta(t)x)\,, \tag{2.10}$$

for adjustable and real α, β and θ, functions of time only. If $\alpha = \sqrt{2\Omega}$, $\theta = \Omega t$ and $\beta = \sqrt{\Omega}$, for constant Ω, then we recover the classical one-soliton solutions for the nonlinear Schrödinger equation. Inserting (2.10) into (2.8), we obtain a mechanical system described by

$$S = \int L\,dt \tag{2.11}$$

with the Lagrange function

$$L = L(\theta,\alpha,\beta,\dot\theta,\dot\alpha,\dot\beta) = \frac{\alpha^2}{15}\left(\frac{30\dot\theta}{\beta} - \frac{10\alpha^2}{\beta} + 10\beta + 8H^2\alpha^2\beta + 14H^2\beta^3\right)\,. \tag{2.12}$$

Proceeding to the variational derivatives, we obtain

$$\frac{\delta S}{\delta\theta} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\alpha^2}{\beta}\right) = 0\,, \tag{2.13}$$

$$\frac{\delta S}{\delta\alpha} = 0 \quad \Rightarrow \quad \alpha\left(\dot\theta - \frac{2\alpha^2}{3} + \frac{\beta^2}{3} + \frac{8H^2\alpha^2\beta^2}{15} + \frac{7H^2\beta^4}{15}\right) = 0\,, \tag{2.14}$$

$$\frac{\delta S}{\delta\beta} = 0 \quad \Rightarrow \quad \alpha^2\left(\dot\theta - \frac{\alpha^2}{3} - \frac{\beta^2}{3} - \frac{4H^2\alpha^2\beta^2}{15} - \frac{7H^2\beta^4}{5}\right) = 0\,. \tag{2.15}$$

The solution for (2.13) is

$$\alpha^2 = 2\sqrt{\Omega}\beta\,, \tag{2.16}$$

where Ω is a numerical constant. Excluding the trivial case $\alpha = 0$ and inserting (2.16) into (2.14), we derive

$$\dot\theta = \frac{4\sqrt{\Omega}\beta}{3} - \frac{\beta^2}{3} - \frac{16\sqrt{\Omega}H^2\beta^3}{15} - \frac{7H^2\beta^4}{15}\,, \tag{2.17}$$

which, taking into account (2.15), gives

$$H^2\beta^3 + \frac{6\sqrt{\Omega}H^2\beta^2}{7} + \frac{5\beta}{14} - \frac{5\sqrt{\Omega}}{14} = 0\,. \tag{2.18}$$

Equation (2.18) is the key equation for our variational approach for the semi-classical Zakharov system in the adiabatic limit. A simplification is achieved through the rescaling

$$\bar\beta = \frac{\beta}{\sqrt{\Omega}}\,, \quad \bar H = \sqrt{\Omega}H\,, \tag{2.19}$$

which eliminates one irrelevant free parameter of (2.18),

$$\bar{H}^2\bar{\beta}^3 + \frac{6\bar{H}^2\bar{\beta}^2}{7} + \frac{5\bar{\beta}}{14} - \frac{5}{14} = 0\,. \tag{2.20}$$

In the formal classical limit $\bar{H} = 0$, one obtain the solution $\bar{\beta} = 1$, which reproduces the classical one-soliton solution. In addition, it can be shown that, for

$$\bar{H}^2 \leq \frac{5}{1152}(681 + 23\sqrt{897}) \approx 5.946\,, \tag{2.21}$$

(2.20) admits just one real root besides two complex roots. This range of parameters seems to be in accordance with the semi-classical limit we have taken, but notice that, as $\bar{H}$ also depends on Ω (see (2.19)), there is no constraint on the maximum value of $\bar{H}$, for sufficiently high Ω (provided $H \neq 0$). Hence, we consider separately both cases.

2.1 Small $\bar{H}^2$

In this situation, (2.20) have the perturbative solution

$$\bar{\beta} = 1 - \frac{26\bar{H}^2}{5} + O(\bar{H}^4)\,. \tag{2.22}$$

Retaining only the terms up to $\bar{H}^2$ and coming back to the original variables, we get

$$\alpha = \sqrt{2\Omega}(1 - \frac{13\Omega H^2}{5})\,, \tag{2.23}$$

$$\beta = \sqrt{\Omega}(1 - \frac{26\Omega H^2}{5})\,, \tag{2.24}$$

$$\dot{\theta} = \Omega - 5\Omega^2 H^2\,. \tag{2.25}$$

Looking at the variational solution (2.10), we conclude that, for small $\bar{H}$, the quantum effects contribute to decrease the amplitude (proportional to α), to enlarge the spatial dispersion (proportional to $1/\beta$) and to decrease the rate of change of the phase (proportional to $\dot{\theta}$) of the classical soliton. All that is in accordance with the dispersive character of quantum effects in the semi-classical limit. They contribute to a smearing of the soliton, perturbing (perhaps in a decisive way) the balance between nonlinearity and dispersion responsible for the existence of soliton solutions for the nonlinear Schrödinger equation.

2.2 Large $\bar{H}^2$

Provided $H \neq 0$, we can access $\bar{H} = \sqrt{\Omega}H \gg 1$ for sufficiently large Ω, even for the semi-classical $H \ll 1$ case we are considering. For large $\bar{H}$, instead of just one class of solutions as in the preceding subsection, we found three subclasses of solutions, namely

$$\bar{\beta}_0 = -\frac{6}{7} + \frac{65}{72}\bar{H}^{-2} + O\left(\bar{H}^{-4}\right)\,, \tag{2.26}$$

$$\bar{\beta}_+ = (\frac{5}{12})^{1/2}\bar{H}^{-1} - \frac{65}{144}\bar{H}^{-2} + O\left(\bar{H}^{-3}\right)\,, \tag{2.27}$$

$$\bar{\beta}_- = -(\frac{5}{12})^{1/2}\bar{H}^{-1} - \frac{65}{144}\bar{H}^{-2} + O\left(\bar{H}^{-3}\right)\,. \tag{2.28}$$

Notice that the last two solutions come from a $1/\bar{H}$ expansion, while the first comes from a $1/\bar{H}^2$ expansion. The accuracy of these perturbative solution can be easily checked by taking some fixed large value of $\bar{H}^2$, solving (2.20) and comparing with (2.26-2.27).

In view of (2.16) and retaining only the leading order term, we obtain, for (2.26) and the solution with negative sign in (2.28), a purely imaginary solution α. This contradicts the proposal (2.10), build with real functions α, β and θ. Hence we discard these solutions and consider only the solution $\bar{\beta}_+$ in (2.27). This gives, considering only the leading order terms,

$$\alpha = \left(\frac{5\Omega}{3H^2}\right)^{1/4}, \quad \beta = \frac{(5/12)^{1/2}}{H}, \quad \dot{\theta} = \frac{4}{9}\sqrt{\frac{5\Omega}{3H^2}}\,. \tag{2.29}$$

Since $H \ll 1$ and $\Omega \gg 1$, this corresponds to a large-amplitude, highly localized and highly oscillating variational solution, with no classical correspondence.

3. Conclusion

Using a variational principle, we have shown that, in the adiabatic and semi-classical limit, quantum effects produce an tendency for smearing out one-soliton solutions for the quantum Zakharov equations. This is not an obvious result at all, specially if we realize that quantum terms appear also in a nonlinear way in the right-hand side of equation (2.7). In addition, we have found a new variational solution, with no classical counterpart. However, we admit that the nonlinear wave solutions for the quantum Zakharov equations are still only poorly understood. For instance, open questions are related to the influence of quantum effects in N-soliton solutions and its stability. In the classical case, numerical experiments [6]-[8] show typically that the solutions for the Zakharov equations relax asymptotically to N-soliton solutions. At this time, no similar numerical investigation have been made for the quantum Zakharov equations.

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