Noether symmetries for two-dimensional charged particle motion

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Abstract

We find the Noether point symmetries for non–relativistic two-dimensional charged particle motion. These symmetries are composed of a quasi–invariance transformation, a time–dependent rotation and a time–dependent spatial translation. The associated electromagnetic field satisfy a system of first–order linear partial differential equations. This system is solved exactly, yielding three classes of electromagnetic fields compatible with Noether point symmetries. The corresponding Noether invariants are derived and interpreted.

1 Introduction

There exist several methods for the derivation of exact invariants (constants of motion or first integrals) for dynamical systems [1]. Among these methods, special attention has been focused on the use of Noether's theorem [2, 3] due to its physical appeal. Noether's theorem stablishes a link between the continuous symmetries of the action functional associated to a dynamical system and its conservation laws. The classical examples of Noether's theorem are the conservation of energy associated to time translation invariance, the conservation of linear momentum associated to space translation invariance and the conservation of angular momentum associated to invariance under rotations.

In the present work, we investigate the Noether point symmetries for two-dimensional non-relativistic charged particle motion. This class of systems is described by Lagrangians of the form

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + A_1(x, y, t)\dot{x} + A_2(x, y, t)\dot{y} - V(x, y, t)\,, \tag{1}$$

where, in appropriated units, $\mathbf{A} = (A_1(x, y, t), A_2(x, y, t), 0)$ is the vector potential and $V(x, y, t)$ is the scalar potential. The results presented here can be useful in the derivation of exact time–dependent solutions of the Vlasov–Maxwell equations in plasma physics [4].

In the course of the search for Noether conserved quantities, we find that the general form of Noether point symmetries for Lagrangians of the class (1) comprises only time-dependent rescalings, rotations and spatial translations. Moreover, the associated electromagnetic fields are constrained by a pair of linear, first-order partial differential equations. The form of this pair of equations depends on the symmetry considered. Nevertheless, for the solution of these equations, we may always resort to the use of canonical group coordinates, defined in section 3. Canonical group coordinates are, in fact, a valuable tool for systematic determination of the general solution of the basic system of equations satisfied by the electromagnetic fields compatible with Noether's point symmetries. We find three categories of such electromagnetic fields, one corresponding to an energy-like constant of motion, one to an angular momentum-like constant of motion and, finally, one with an associated linear momentum-like first integral. These conserved quantities are gauge independent, a fact that is explicitly shown in all cases treated here. This shows that the choice of gauge has no influence in the determination of Noether's conserved quantities, in contrary to what is sometimes claimed [5].

The paper is organized as follows. In section 2, we present Noether's theorem for the case of point symmetries. The condition for invariance of the action functional with a Lagrangian of type (1) yields both the form of the Noether symmetries and the equations to be satisfied by the electromagnetic field. These equations, listed in (34–36) below, are the basic equations to be satisfied by the electromagnetic field in order to exist the corresponding Noether point symmetries. The system (34–36) is solved by use of canonical group variables for each type of Noether symmetry. These canonical group variables are shown in section 3. The corresponding solutions to system (34–36) is given in section 4. In order that the solutions so obtained qualify as true electromagnetic fields, the Maxwell equations are imposed for consistency. In section 5, the Noether invariant associated to each symmetry is showed and interpreted. Finally, section 6 is devoted to the conclusions.

2 Noether point symmetries

Noether's theorem provides a link between continuous symmetries for the action functional

$$S = \int L \, dt \tag{2}$$

and conserved quantities. In its original and more powerful formulation [2, 3], Noether's theorem considers dynamic symmetries involving velocities and higher derivatives. Here, mainly for simplifying reasons, we restrict considerations to point transformations. In more formal terms, we consider infinitesimal mappings of the form

$$\bar{x} = x + \varepsilon\eta_1(x, y, t) \,, \tag{3}$$

$$\bar{y} = y + \varepsilon\eta_2(x, y, t) \,, \tag{4}$$

$$\bar{t} = t + \varepsilon\tau(x, y, t) \,, \tag{5}$$

where ε is an infinitesimal parameter. It is useful to define the generator of the group of symmetries associated to (3–5) as

$$G = \tau\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial x} + \eta_2\frac{\partial}{\partial y} \,. \tag{6}$$

The generator G appear frequently in what follows and it is useful in the definition of canonical group coordinates, which plays a central role in the systematic determination of the electromagnetic fields associated to the symmetries.

The condition for Noether symmetry reads [3]

$$\tau\, L_t + \boldsymbol{\eta} \cdot\, L_{\mathbf{q}} + (\dot{\boldsymbol{\eta}} - \dot{\tau}\dot{\mathbf{q}}) \cdot\, L_{\dot{\mathbf{q}}} + \dot{\tau}\, L = \dot{F}\,, \tag{7}$$

where subscripts denote partial derivatives. In equation (7) and in what follows,

$$\boldsymbol{\eta} \;=\; (\eta_1, \eta_2, 0)\,, \tag{8}$$

$$\mathbf{q} \;=\; (x, y, 0)\,, \tag{9}$$

$$F \;=\; F(x, y, t)\,. \tag{10}$$

The condition for Noether symmetry ensures the invariance of the action functional under the infinitesimal map (3–5), up to the addition of an irrelevant numerical constant. Now, as a consequence, the Euler-Lagrange equations become formally invariant under the map (3–5) and the associated Noether invariant reads

$$I = \tau\, (\dot{\mathbf{q}} \cdot\, L_{\dot{\mathbf{q}}} - L) - \boldsymbol{\eta} \cdot\, L_{\dot{\mathbf{q}}} + F\,, \tag{11}$$

which, in the present case, has the form

$$I = \tau \left(\frac{1}{2}\dot{\mathbf{q}}^2 + V\right) - \boldsymbol{\eta} \cdot (\dot{\mathbf{q}} + \mathbf{A}) + F\,. \tag{12}$$

Clearly, for point symmetries, the Noether invariant is at most quadratic in the velocity. This reminds of the work of Lewis [6], where he finds quadratic constants of motion for the three–dimensional non–relativistic motion of a charged particle in an external electromagnetic field. The work of Lewis, however, did not obtain the whole variety of solutions. Some of the Noether invariants derived in section 4 of this paper do not fit into the framework of reference [6].

In what follows, we find all the Noether point symmetries associated to Lagrangian (1). The Noether point symmetries are identified by imposing the symmetry condition (7) and the Noether invariants are constructed by use of equation (12).

We begin by identifying the Noether point symmetries associated with the Lagrangian (31). Inserting L in Noether's symmetry condition (7), we arrive at a polynomial in the velocity components. The coefficients of all monomials of the form $\dot{x}^m \dot{y}^n$ must be identically zero. For instance, the coefficients of $\dot{x}^3$ and $\dot{y}^3$ yield

$$\dot{x}^3 \rightarrow \tau_x = 0\,, \tag{13}$$

$$\dot{y}^3 \rightarrow \tau_y = 0\,. \tag{14}$$

The solution for (13–14) is

$$\tau = \rho^2(t)\,, \tag{15}$$

$\rho(t)$ an arbitrary function of time. Equation (15) is taken into account in the continuation.

The coefficients of $\dot{x}^2\dot{y}$ and $\dot{x}\dot{y}^2$ give no new information, as they are identically zero. The coefficients of the quadratic components of velocity yield

$$\dot{x}^2 \rightarrow \eta_{1x} = \rho\dot{\rho}\,, \tag{16}$$

$$\dot{y}^2 \rightarrow \eta_{2y} = \rho\dot{\rho}\,, \tag{17}$$

$$\dot{x}\dot{y} \rightarrow \eta_{1y} + \eta_{2x} = 0\,. \tag{18}$$

The general solution of (18) is

$$\eta_1 = \Gamma_x \quad , \quad \eta_2 = -\Gamma_y \quad , \tag{19}$$

where $\Gamma = \Gamma(x, y, t)$ is an arbitrary function. Inserting eq. (19) into eqs. (16–17), yield

$$\Gamma_{xx} = -\Gamma_{yy} = \rho\dot{\rho}\,. \tag{20}$$

The solution to equation (20) is a quadratic function of the spatial coordinates,

$$\Gamma = \frac{\rho\dot{\rho}}{2}(x^2 - y^2) - \Omega(t)\,xy + a_1(t)\,x - a_2(t)\,y + \Gamma_0(t)\,, \tag{21}$$

for arbitrary functions of time $\Omega(t)$, $a_1(t)$, $a_2(t)$ and $\Gamma_0(t)$.

By inserting the solution (21) into eq. (19), we obtain

$$\eta_1 = \rho\dot{\rho}\,x - \Omega(t)\,x + a_1(t)\,, \tag{22}$$

$$\eta_2 = \rho\dot{\rho}\,y + \Omega(t)\,y + a_2(t)\,. \tag{23}$$

Let us summarize our first results: up to terms quadratic in the velocity, the generator of Noether point symmetries has the general form

$$G = G_Q + G_R + G_T\,, \tag{24}$$

where

$$G_Q = \rho^2(t)\frac{\partial}{\partial t} + \rho\dot{\rho}\,(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) \quad , \tag{25}$$

is associated to quasi–invariance transformations [7],

$$G_R = \Omega(t)\,(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \tag{26}$$

is associated to time–dependent rotations and

$$G_T = a_1(t)\frac{\partial}{\partial x} + a_2(t)\frac{\partial}{\partial y} \tag{27}$$

corresponds to time–dependent spatial translations. It is also important to notice that, so far, no constraint has been imposed on the electromagnetic fields.

The Noether symmetry condition, however, has not yet been fully taken into account. We still must consider the terms linear in the velocity components, yielding

$$\dot{x} \to F_x = G\,A_1 + \rho\dot{\rho}\,A_1 + \Omega\,A_2 + (\rho\ddot{\rho} + \dot{\rho}^2)\,x - \dot{\Omega}\,y + \dot{a}_1\,, \tag{28}$$

$$\dot{y} \to F_y = G\,A_2 + \rho\dot{\rho}\,A_2 - \Omega\,A_1 + (\rho\ddot{\rho} + \dot{\rho}^2)\,y + \dot{\Omega}\,x + \dot{a}_2\,. \tag{29}$$

Moreover, the term independent of velocities gives

$$\begin{aligned} F_t &= -G\,V - 2\rho\dot{\rho}\,V + \Big((\rho\ddot{\rho} + \dot{\rho}^2)\,x - \dot{\Omega}\,y + \dot{a}_1\Big)\,A_1 + \\ &+ \Big((\rho\ddot{\rho} + \dot{\rho}^2)\,y + \dot{\Omega}\,x + \dot{a}_2\Big)\,A_2\,. \end{aligned} \tag{30}$$

The system (28–30) has a solution $F(x,y,t)$ if and only if the integrability conditions $F_{xy} = F_{yx}$, $F_{xt} = F_{tx}$ and $F_{yt} = F_{ty}$ are fulfilled. These requirements give equations not for the electromagnetic potentials, but directly for the electric field $\mathbf{E} = (E_1(x,y,t), E_2(x,y,t), 0)$ and the magnetic field $\mathbf{B} = (0, 0, B(x,y,t))$, defined by

$$E_1(x,y,t) = -V_x - A_{1t}\,, \tag{31}$$

$$E_2(x,y,t) = -V_y - A_{2t}\,, \tag{32}$$

$$B(x,y,t) = A_{2x} - A_{1y}\,. \tag{33}$$

In fact, imposing $F_{xy} = F_{yx}$ yields

$$G\,B = -2\rho\dot{\rho}\,B - 2\dot{\Omega}\,, \tag{34}$$

which involves only the magnetic field. Imposition of $F_{xt} = F_{tx}$ implies

$$\begin{aligned} G\,E_1 &= -3\rho\dot{\rho}\,E_1 - \Omega\,E_2 - \Big((\rho\ddot{\rho} + \dot{\rho}^2)y + \dot{\Omega}\,x + \dot{a}_2\Big)\;B + \\ &+ \;(\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho})\,x - \ddot{\Omega}\,y + \ddot{a}_1\,, \end{aligned} \tag{35}$$

whereas $F_{yt} = F_{ty}$ implies

$$\begin{aligned} G\,E_2 &= -3\rho\dot{\rho}\,E_2 + \Omega\,E_1 + \Big((\rho\ddot{\rho} + \dot{\rho}^2)x - \dot{\Omega}\,y + \dot{a}_1\Big)\;B + \\ &+ \;(\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho})\,y + \ddot{\Omega}\,x + \ddot{a}_2\,. \end{aligned} \tag{36}$$

Equations (34–36) are the equations to be satisfied by the electromagnetic fields associated to Noether point symmetries. They constitute a system of linear, first order, coupled partial differential equations for E_1, E_2 and B. For each non-relativistic charged particle motion under an electromagnetic field satisfying (34–36), there is an associated Noether point symmetry, whose generator is given by eq. (24). In the remaining of this work, we are essentially interested in finding all the solutions of the system of partial differential equations (34–36). In other words, we are concerned with finding the most general electromagnetic field for two-dimensional, non-relativistic charged particle motion endowed with Noether point symmetries. It is interesting to note that, while Noether's theorem is formulated in terms of the action functional, which is gauge dependent, the final conditions for Noether point symmetry involves only the physical fields and the symmetry generator and not the potentials. This means that the choice of gauge does not play any role in the search for Noether point symmetries. Another useful remark is that B satisfies an equation decoupled from the equations for E_1 and E_2, whereas the equations for the electric field do depend on B. Thus, we must first solve (34) for B and afterwards solve (35–36) for the electric field.

The unknown functions are the electromagnetic potentials $\mathbf{A}$ and V. Therefore, for any solution $\mathbf{E}$, $\mathbf{B}$ of the system (34–36), we still have to solve equations (31–33) to obtain the electromagnetic potentials. It turns out that the integrability condition for the system (31–33) are the homogeneous Maxwell equations. Gauss law for magnetism is immediately satisfied here. So, the only extra requirement that we must impose is Faraday's law

$$E_{2x} - E_{1y} + B_t = 0\,. \tag{37}$$

With that last constraint, the solutions $\mathbf{E}$, $\mathbf{B}$ for the basic system (34–36) qualifies as a true electromagnetic field.

To treat the system (34–36) and to find its complete solution, we use canonical group coordinates. These variables are be introduced in the section that follows.

3 Canonical group coordinates

Canonical group coordinates [8] are defined by imposing that the symmetry transformation behave merely like time translation. Denoting new coordinates by $(\bar{x}, \bar{y}, \bar{t})$, such condition means that, in canonical group coordinates,

$$G = \frac{\partial}{\partial \bar{t}}\,, \tag{38}$$

where $\bar{t}$ is the new time parameter. The equations which must be satisfied by any set of canonical group coordinates are given by

$$G\,\bar{x} = 0 \quad , \quad G\,\bar{y} = 0 \quad , \quad G\,\bar{t} = 1 \quad . \tag{39}$$

This is a set of uncoupled linear partial differential equations, which, for the generator (24), can be solved in closed form by the method of characteristics. We find three classes of solutions, listed below.

3.1 The case $\rho \neq 0$

When $\rho \neq 0$, it is convenient to write

$$a_1 = \rho(\rho\dot{\alpha}_1 - \dot{\rho}\alpha_1)\,, \tag{40}$$

$$a_2 = \rho(\rho\dot{\alpha}_2 - \dot{\rho}\alpha_2) \tag{41}$$

for suitable functions $\alpha_1(t)$ and $\alpha_2(t)$, defined in terms of a_1 and a_2. Notice that for $\rho = 0$ the transformation (40–41) is meaningless and the case is treated separately.

With the redefinition (40–41), we have the following canonical group coordinates,

$$\bar{t} = \int^t d\mu/\rho^2(\mu)\,, \tag{42}$$

$$\bar{x} = \frac{(x-\alpha_1)}{\rho}\cos T + \frac{(y-\alpha_2)}{\rho}\sin T + \delta_1\,, \tag{43}$$

$$\bar{y} = \frac{(y-\alpha_2)}{\rho}\cos T - \frac{(x-\alpha_1)}{\rho}\sin T + \delta_2\,, \tag{44}$$

where new functions $T = T(t)$, $\delta_1 = \delta_1(t)$ and $\delta_2 = \delta_2(t)$ were defined according to

$$
\begin{aligned}
T(t) &= \int^t d\mu\, \Omega(\mu)/\rho^2(\mu)\,, && (45)\\
\delta_1(t) &= \int^t d\mu \frac{\Omega(\mu)}{\rho^3(\mu)} \left(\alpha_2(\mu)\cos T(\mu) - \alpha_1(\mu)\sin T(\mu)\right)\,, && (46)\\
\delta_2(t) &= -\int^t d\mu \frac{\Omega(\mu)}{\rho^3(\mu)} \left(\alpha_1(\mu)\cos T(\mu) + \alpha_2(\mu)\sin T(\mu)\right)\,. && (47)
\end{aligned}
$$

As a particular case, let us consider the situation where the symmetry transformation does not contain rotation, that is, $\Omega = 0$. In that case, the canonical group variables (42–44) take the form

$$
\begin{aligned}
\bar{t} &= \int^t d\mu/\rho^2(\mu)\,, && (48)\\
\bar{x} &= \frac{(x-\alpha_1)}{\rho}\,, && (49)\\
\bar{y} &= \frac{(y-\alpha_2)}{\rho}\,, && (50)
\end{aligned}
$$

which are relevant in the study of time–dependent integrable systems [9]. For $\alpha_1 = \alpha_2 = 0$, the transformation (48–50) is known as a quasi-invariance transformation [7].

3.2 The case $\rho = 0$ and $\Omega \neq 0$

The canonical group variables become, in this case,

$$
\begin{aligned}
\bar{x} &= \left((x-\beta_1)^2 + (y-\beta_2)^2\right)^{1/2}\,, && (51)\\
\bar{y} &= t\,, && (52)\\
\bar{t} &= \frac{1}{\Omega}\tan^{-1}\left(\frac{y-\beta_2}{x-\beta_1}\right)\,, && (53)
\end{aligned}
$$

where we have defined

$$\beta_1 = \beta_1(t) = -a_2/\Omega \quad , \quad \beta_2 = \beta_2(t) = a_1/\Omega \quad . \tag{54}$$

The variables $\bar{x}$ and $\bar{t}$ are translated polar coordinates, the new time parameter plays the role of an azimuthal angle and $\bar{x}$ plays the role of a radial coordinate.

3.3 The case $\rho = 0$, $\Omega = 0$ and $a_2 \neq 0$

The canonical group variables are now

$$\bar{x} = x - a_1 y/a_2 \, , \tag{55}$$

$$\bar{y} = t \, , \tag{56}$$

$$\bar{t} = y/a_2 \, . \tag{57}$$

We finally mention that the case $\rho = 0$, $\Omega = 0$ and $a_1 \neq 0$ is strictly analogous to the last case and deserve no special consideration.

In the following section, we obtain all the solutions for the basic system (34–36), corresponding to each set of canonical group variables.

4 Electromagnetic fields

4.1 The case $\rho \neq 0$

Equation (34), which involves only the magnetic field acquires, in canonical group coordinates, the form

$$B_{\bar{t}} = -\frac{2\rho'}{\rho} B - \frac{2\Omega'}{\rho^2} \, , \tag{58}$$

where the prime denotes total differentiation with respect to $\bar{t}$. The solution for (58) is

$$B = -\frac{2\Omega}{\rho^2} + \frac{1}{\rho^2}\bar{B}(\bar{x},\bar{y})\,, \tag{59}$$

where $\bar{B}(\bar{x},\bar{y})$ is an arbitrary function of the indicated arguments. Notice that the resulting magnetic field is not necessarily homogeneous, since it can depend on the spatial coordinates through $\bar{x}$ and $\bar{y}$. This is a significant improvement on earlier results [10].

To find the electric field, we must solve the system (35–36), taking the solution (59) into account. To solve (35–36), it is useful to consider the quantities Σ_1 and Σ_2 defined by

$$\Sigma_1 = \rho^3(E_1\cos T + E_2\sin T)\,, \tag{60}$$

$$\Sigma_2 = \rho^3(E_2\cos T - E_1\sin T)\,. \tag{61}$$

This represents a rotation plus a rescaling of the electric field. The form (60–61) represents a circularly polarized wave with time-dependent amplitude. .

In the new variables, the system (35–36) decouples and can be cast into the form

$$\frac{\partial\Sigma_1}{\partial\bar{t}} = \frac{\partial\psi_1}{\partial\bar{t}} \quad , \quad \frac{\partial\Sigma_2}{\partial\bar{t}} = \frac{\partial\psi_2}{\partial\bar{t}} \quad , \tag{62}$$

where

$$\begin{aligned}\psi_1 \;=\; & \left(-\frac{\rho'}{\rho}(\bar{y}-\delta_2) + \delta_2' - \Omega(\bar{x}-\delta_1) + \frac{1}{\rho}(\alpha_1'\sin T - \alpha_2'\cos T)\right)\bar{B}(\bar{x},\bar{y}) + \\ + \; & \left(\frac{\rho''}{\rho} - 2\frac{\rho'^2}{\rho^2} + \Omega^2\right)(\bar{x}-\delta_1) - \left(\Omega' - 2\frac{\rho'}{\rho}\Omega\right)(\bar{y}-\delta_2) + \\ + \; & \frac{1}{\rho}\left(\Omega'\alpha_1 - \Omega(\alpha_1' + \frac{\rho'}{\rho}\alpha_1) + \alpha_2'' - 2\frac{\rho'}{\rho}\alpha_2' + \Omega^2\alpha_2\right)\sin T +\end{aligned} \tag{63}$$

$$+ \quad \frac{1}{\rho}\left(-\Omega'\alpha_2 + \Omega(\alpha_2' + \frac{\rho'}{\rho}\alpha_2) + \alpha_1'' - 2\frac{\rho'}{\rho}\alpha_1' + \Omega^2\alpha_1\right)\cos T\,,$$

$$\begin{aligned}
\psi_2 \;=\; & \left(+\frac{\rho'}{\rho}(\bar{x}-\delta_1) - \delta_1' - \Omega(\bar{y}-\delta_2) + \frac{1}{\rho}(\alpha_1'\cos T + \alpha_2'\sin T)\right)\bar{B}(\bar{x},\bar{y}) + \\
+ \; & \left(\frac{\rho''}{\rho} - 2\frac{\rho'^2}{\rho^2} + \Omega^2\right)(\bar{y}-\delta_2) + \left(\Omega' - 2\frac{\rho'}{\rho}\Omega\right)(\bar{x}-\delta_1) + \\
- \; & \frac{1}{\rho}\left(-\Omega'\alpha_2 + \Omega(\alpha_2' + \frac{\rho'}{\rho}\alpha_2) + \alpha_1'' - 2\frac{\rho'}{\rho}\alpha_1' + \Omega^2\alpha_1\right)\sin T + \qquad (64) \\
+ \; & \frac{1}{\rho}\left(+\Omega'\alpha_1 - \Omega(\alpha_1' + \frac{\rho'}{\rho}\alpha_1) + \alpha_2'' - 2\frac{\rho'}{\rho}\alpha_2' + \Omega^2\alpha_2\right)\cos T\,.
\end{aligned}$$

The solution to (62) is

$$\Sigma_1 = \psi_1 + \bar{E}_1(\bar{x},\bar{y}) \quad , \quad \Sigma_2 = \psi_2 + \bar{E}_2(\bar{x},\bar{y})\,, \qquad (65)$$

where, as indicated, $\bar{E}_1$ and $\bar{E}_2$ have no dependence on $\bar{t}$.

We are interested in the electric field, in physical variables. To obtain the physical field we use the inverse of the transformation (60–61),

$$E_1 = \frac{1}{\rho^3}(\Sigma_1\cos T - \Sigma_2\sin T)\,, \qquad (66)$$

$$E_2 = \frac{1}{\rho^3}(\Sigma_2\cos T + \Sigma_1\sin T)\,. \qquad (67)$$

Substituting equations (66–67) into the solution (65) and transforming back to the original variables (x, y, t), we obtain the electric fields

$$\begin{aligned}
E_1 \;=\; & \ddot{\alpha}_1 + \frac{\ddot{\rho}}{\rho}(x-\alpha_1) + \frac{\Omega^2 x}{\rho^4} - (\rho\dot{\Omega} - 2\dot{\rho}\Omega)\frac{y}{\rho^3} + \frac{\Omega}{\rho^3}(\rho\dot{\alpha}_2 - \dot{\rho}\alpha_2) + \\
+ \; & \frac{1}{\rho^3}\left(\bar{E}_1(\bar{x},\bar{y})\cos T - \bar{E}_2(\bar{x},\bar{y})\sin T\right) \qquad (68) \\
- \; & \frac{1}{\rho^4}\left(\rho\dot{\rho}(y-\alpha_2) + \rho^2\dot{\alpha}_2 + \Omega\, x\right)\bar{B}(\bar{x},\bar{y})\,,
\end{aligned}$$

$$
\begin{aligned}
E_2 &= \ddot{\alpha}_2 + \frac{\ddot{\rho}}{\rho}(y-\alpha_2) + \frac{\Omega^2 y}{\rho^4} + (\rho\dot{\Omega} - 2\dot{\rho}\Omega)\frac{x}{\rho^3} - \frac{\Omega}{\rho^3}(\rho\dot{\alpha}_1 - \dot{\rho}\alpha_1) + \\
&+ \frac{1}{\rho^3}\Big(\bar{E}_2(\bar{x},\bar{y})\cos T + \bar{E}_1(\bar{x},\bar{y})\sin T\Big) + \\
&+ \frac{1}{\rho^4}\Big(\rho\dot{\rho}(x-\alpha_1) + \rho^2\dot{\alpha}_1 - \Omega\, y\Big)\,\bar{B}(\bar{x},\bar{y})\,.
\end{aligned} \tag{69}
$$

It still remains to take into consideration Faraday's law, which, in our case, is equivalent to eq. (37). After lengthy calculations using the magnetic field (59) and the electric field (68–69), we conclude that Faraday's law imposes only that

$$
\bar{E}_{2\bar{x}} - \bar{E}_{1\bar{y}} = 0\,. \tag{70}
$$

This last constraint is an equation that has general solution

$$
\bar{E}_1 = -\frac{\partial}{\partial\bar{x}}\bar{V}(\bar{x},\bar{y}) \quad , \quad \bar{E}_2 = -\frac{\partial}{\partial\bar{y}}\bar{V}(\bar{x},\bar{y}) \quad , \tag{71}
$$

where $\bar{V}(\bar{x},\bar{y})$ is an arbitrary function of the indicated argument.

In conclusion, we have obtained a very general class of electromagnetic fields yielding Noether point symmetries. The magnetic field is given by eq. (59) and the electric field by eqs. (68–69), together with condition (71). The symmetry transformations has the generator (24). The electromagnetic field involves several arbitrary functions, namely $\rho(t), \alpha_1(t), \alpha_2(t), \Omega(t), \bar{B}(\bar{x},\bar{y})$ and $\bar{V}(\bar{x},\bar{y})$, where $\bar{x}$ and $\bar{y}$ are defined by eqs. (43–44).

Finally, the electric field may be represented in a much more compact way. Introducing the vectors

$$
\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0)\,, \tag{72}
$$

$$
\boldsymbol{\Omega} = (0, 0, \Omega)\,, \tag{73}
$$

$$\bar{\mathbf{E}} = (\bar{E}_1, \bar{E}_2, 0)\,, \tag{74}$$

$$\bar{\mathbf{B}} = (0, 0, \bar{B})\,, \tag{75}$$

redefining $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = \rho\dot{\rho}(\mathbf{q} - \boldsymbol{\alpha}) + \rho^2\dot{\boldsymbol{\alpha}} + \boldsymbol{\Omega} \times \mathbf{q}\,, \tag{76}$$

and using the rotation matrix

$$R(T) = \begin{pmatrix} \cos T & -\sin T & 0 \\ \sin T & \cos T & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \tag{77}$$

we can represent the electric field by the form

$$\mathbf{E} = \frac{1}{\rho^4}(\rho(\rho\boldsymbol{\eta}_t - \dot{\rho}\boldsymbol{\eta}) + \boldsymbol{\eta} \times \boldsymbol{\Omega}) + \frac{1}{\rho^3}R(T)\bar{\mathbf{E}} + \frac{1}{\rho^4}\bar{\mathbf{B}} \times \boldsymbol{\eta}\,, \tag{78}$$

where $\bar{\mathbf{E}}$ is given in terms of a scalar potential according to (71).

4.2 The case $\rho = 0$ and $\Omega \neq 0$

In the case $\rho = 0$, the symmetry transformation is composed of a rotation and a spatial translation and has no rescaling part. The treatment presented in the last subsection is no longer appropriate, because the limit $\rho = 0$ of the canonical variables (42–44) is singular. We must now use the canonical group variables (51–53). The steps to be carried out comprise: a) the calculation of B using the basic equation (34), b) the calculation of E_1 and E_2 using the basic equations (35–36), and c) the imposition of Faraday's law, which must be obeyed by the resulting electromagnetic field.

The equation (34) for B, in canonical group variables (51–53), reads

$$B_{\bar{t}} = -2\dot{\Omega}(\bar{y})\,, \tag{79}$$

having the general solution

$$B = -2\dot{\Omega}(\bar{y})\bar{t} + \bar{B}(\bar{x}, \bar{y})\,, \qquad (80)$$

where $\bar{B}(\bar{x}, \bar{y})$ is an arbitrary function of the indicated arguments.

By the definition (53) of the new time parameter, however, it is clear that $\bar{t}$ is not a single valued function. Thus, the resulting expression B in eq. (80) is not well behaved if $\dot{\Omega} \neq 0$. Consequently, in order to stay with a physically meaningful result, we must impose the constraint

$$\dot{\Omega} = 0\,. \qquad (81)$$

Without any loss of generality, we can also take

$$\Omega = 1\,. \qquad (82)$$

The associated Noether symmetries now comprises a time–independent rotation and two time–dependent spatial translations. The associated magnetic field, according to the solution (80) and the restriction (81), has the form

$$B = \bar{B}(\bar{x}, \bar{y})\,. \qquad (83)$$

Putting this functional form B into the system (35–36), taking into account $\rho = 0$, $\Omega = 1$ and the definition (54) of β_1 and β_2, yields

$$E_{1\bar{t}} = \ddot{\beta}_2 - E_2 + \dot{\beta}_1 \bar{B}(\bar{x}, \bar{y})\,, \qquad (84)$$

$$E_{2\bar{t}} = -\ddot{\beta}_1 + E_1 + \dot{\beta}_2 \bar{B}(\bar{x}, \bar{y})\,. \qquad (85)$$

In these equations, $\bar{x}$ and $\bar{y}$ are formally parameters independent of $\bar{t}$. Consequently, it is not difficult to obtain the solution, which reads

$$E_1 = \ddot{\beta}_1 - \dot{\beta}_2 \bar{B}(\bar{x}, \bar{y}) +$$

$$
\begin{aligned}
&+ \quad \tilde{E}_1(\bar{x},\bar{y})\cos\bar{t} - \tilde{E}_2(\bar{x},\bar{y})\sin\bar{t}\,, && (86)\\
E_2 \quad &= \quad \ddot{\beta}_2 + \dot{\beta}_1 \bar{B}(\bar{x},\bar{y}) + \\
&+ \quad \tilde{E}_2(\bar{x},\bar{y})\cos\bar{t} + \tilde{E}_1(\bar{x},\bar{y})\sin\bar{t}\,, && (87)
\end{aligned}
$$

where $\tilde{E}_1$ and $\tilde{E}_2$ are arbitrary functions of $\bar{x}$ and $\bar{y}$.

By defining new arbitrary functions

$$
\bar{E}_1 = \tilde{E}_1/\bar{x} \quad , \quad \bar{E}_2 = \tilde{E}_2/\bar{x} \quad , \tag{88}
$$

we obtain, in physical coordinates,

$$
\begin{aligned}
E_1 \quad &= \quad \ddot{\beta}_1 - \dot{\beta}_2 \bar{B}(\bar{x},\bar{y}) + \\
&+ \quad (x-\beta_1)\bar{E}_1(\bar{x},\bar{y}) - (y-\beta_2)\bar{E}_2(\bar{x},\bar{y})\,, && (89)\\
E_2 \quad &= \quad \ddot{\beta}_2 + \dot{\beta}_1 \bar{B}(\bar{x},\bar{y}) + \\
&+ \quad (x-\beta_1)\bar{E}_2(\bar{x},\bar{y}) + (y-\beta_2)\bar{E}_1(\bar{x},\bar{y})\,, && (90)
\end{aligned}
$$

where $\bar{x}$, $\bar{y}$ are given in equations (51–52).

Faraday's law now imposes

$$
\bar{x}\bar{E}_{2\bar{x}} + 2\bar{E}_2 = -\bar{B}_{\bar{y}}\,, \tag{91}
$$

whose solution is

$$
\bar{E}_2 = \frac{1}{\bar{x}^2}\frac{\partial\psi}{\partial\bar{y}} \quad , \quad \bar{B} = -\frac{1}{\bar{x}}\frac{\partial\psi}{\partial\bar{x}} \quad , \tag{92}
$$

where $\psi = \psi(\bar{x},\bar{y})$ is an arbitrary function.

In conclusion, the electromagnetic field is given by eqs. (83) and (89–90), with the constraint (92). There remain four arbitrary functions, namely $E_1(\bar{x},\bar{y})$, $\psi(\bar{x},\bar{y})$, $\beta_1(t)$ and $\beta_2(t)$, with $\bar{x}$, $\bar{y}$ defined in equations (51–52).

4.3 The case $\rho = 0$, $\Omega = 0$ and $a_2 \neq 0$

The procedure to find the electromagnetic field is by now clear. We simply list the results. The equation for the magnetic field is

$$B_{\bar{t}} = 0\,, \tag{93}$$

or

$$B = \bar{B}(\bar{x}, \bar{y})\,, \tag{94}$$

with $\bar{B} = \bar{B}(\bar{x}, \bar{y})$ an arbitrary function of $\bar{x}$ and $\bar{y}$, which are defined by equations (56–57).

The equations for E_1 and E_2 are

$$E_{1\bar{t}} = \ddot{a}_1(\bar{y}) - \dot{a}_2(\bar{y})\bar{B}\,, \tag{95}$$

$$E_{2\bar{t}} = \ddot{a}_2(\bar{y}) + \dot{a}_1(\bar{y})\bar{B}\,, \tag{96}$$

with solution

$$E_1 = (\ddot{a}_1(\bar{y}) - \dot{a}_2(\bar{y})\bar{B}(\bar{x}, \bar{y}))\bar{t} + \bar{E}_1(\bar{x}, \bar{y})\,, \tag{97}$$

$$E_2 = (\ddot{a}_2(\bar{y}) + \dot{a}_1(\bar{y})\bar{B}(\bar{x}, \bar{y}))\bar{t} + \bar{E}_2(\bar{x}, \bar{y})\,, \tag{98}$$

where $\bar{E}_1$ and $\bar{E}_2$ are arbitrary functions of the indicated arguments. In physical coordinates,

$$E_1 = \frac{\ddot{a}_1 y}{a_2} - \frac{\dot{a}_2 y}{a_2}\bar{B}(\bar{x}, \bar{y}) + \bar{E}_1(\bar{x}, \bar{y})\,, \tag{99}$$

$$E_2 = \frac{\ddot{a}_2 y}{a_2} + \frac{\dot{a}_1 y}{a_2}\bar{B}(\bar{x}, \bar{y}) + \bar{E}_2(\bar{x}, \bar{y})\,, \tag{100}$$

where $\bar{x}$, $\bar{y}$ defined in equations (55–56).

After solving the differential equations that arise from Noether's symmetry condition, we must verify what is the constraint imposed by Faraday's law. After some calculation using eqs. (94), (99–100) and the form (55–57) of the canonical group coordinates, we conclude that Faraday's law implies

$$\bar{B}_{\bar{y}} + \left(\frac{1}{a_2}(a_1\bar{E}_1 + a_2\bar{E}_2 - \ddot{a}_1\bar{x} + \dot{a}_2 \int^{\bar{x}} B(\mu, \bar{y}) d\mu)\right)_{\bar{x}} = 0 \,. \tag{101}$$

The solution, in terms of an arbitrary function $\psi = \psi(\bar{x}, \bar{y})$, is

$$\bar{B} = \frac{\partial \psi}{\partial \bar{x}} \,, \tag{102}$$

$$\frac{1}{a_2}(a_1\bar{E}_1 + a_2\bar{E}_2 - \ddot{a}_1\bar{x} + \dot{a}_2 \int^{\bar{x}} \bar{B}(\mu, \bar{y}) d\mu) = -\frac{\partial \psi}{\partial \bar{y}} \,, \tag{103}$$

where the last equation can be rewritten in terms of a new function $\bar{V} = \bar{V}(\bar{x}, \bar{y})$ according to

$$\bar{E}_1 = -\bar{V}_{\bar{x}} \,, \tag{104}$$

$$\bar{E}_2 = \frac{\ddot{a}_1}{a_2}\bar{x} - \frac{\dot{a}_2}{a_2}\psi - \psi_{\bar{y}} + \frac{a_1}{a_2}\bar{V}_{\bar{x}} \,. \tag{105}$$

This completely determines the last class of solutions for the electromagnetic field. B is given by eq. (94) and E_1 and E_2 are given by eqs. (99–100). The functions $\bar{B}$, $\bar{E}_1$ and $\bar{E}_2$, present in the solution, are given by eqs. (102) and (104–105), in terms of the arbitrary functions $\psi(\bar{x}, \bar{y})$ and $\bar{V}(\bar{x}, \bar{y})$ with $\bar{x}$, $\bar{y}$ defined in equations (55–56). The arbitrary functions $a_1(t)$ and $a_2(t)$ are also present in the electromagnetic field, so that four arbitrary functions participate in the final solution.

5 Conserved quantities

The electromagnetic potentials, which are gauge dependent, are a basic ingredient in the Noether's invariant (12). However, the resulting Noether's constant of motion is always independent of gauge choice, as seen in the continuation.

To find the Noether constant of motion corresponding to each of the point symmetry, we must also solve equations (28–30) for the function $F(x, y, t)$ which appears in the definition (12). The system (28–30) is solvable for F by construction, since the electromagnetic fields derived in the last section satisfy the basic equations (34–36). These equations, in turn, are the necessary and sufficient conditions for the existence of a solution F, satisfying system (28–30).

Given the electromagnetic fields and the related symmetries, it is not difficult to find the appropriate electromagnetic potentials and the associated function F. We only show the results pertaining to each type of solution.

5.1 The case $\rho \neq 0$

The vector potential for the magnetic field listed in (59) is

$$\mathbf{A} = \frac{\mathbf{q} \times \mathbf{\Omega}}{\rho^2} + \frac{R(T) \cdot \bar{\mathbf{A}}(\bar{x}, \bar{y})}{\rho} + \nabla\lambda \,, \tag{106}$$

for arbitrary gauge function $\lambda = \lambda(x, y, t)$. The gauge function λ is irrelevant in the calculation of the magnetic field. However, it was kept in order to show explicitly the gauge independence of Noether's constant of motion.

Furthermore, $\bar{\mathbf{A}} = (\bar{A}_1(\bar{x},\bar{y}),\ \bar{A}_2(\bar{x},\ \bar{y}),\ 0)$ is a vector satisfying

$$\bar{A}_{2\bar{x}} - \bar{A}_{1\bar{y}} = \bar{B}\,, \tag{107}$$

where $\bar{B}$ is defined in eq. (59).

The scalar potential is

$$\begin{aligned} V &= -(\rho\ddot{\boldsymbol{\alpha}} - \ddot{\rho}\boldsymbol{\alpha})\cdot\frac{\mathbf{q}}{\rho} - \frac{\ddot{\rho}}{2\rho}\mathbf{q}^2 - \frac{1}{\rho^3}(\rho\dot{\boldsymbol{\alpha}} - \dot{\rho}\boldsymbol{\alpha})\times\boldsymbol{\Omega}\cdot\mathbf{q} - \frac{1}{2\rho^4}(\boldsymbol{\Omega}\times\mathbf{q})^2 + \\ &+ \frac{1}{\rho^2}\bar{V}(\bar{x},\bar{y}) + \frac{\boldsymbol{\eta}\cdot\ R(T)\cdot\bar{\mathbf{A}}(\bar{x},\bar{y})}{\rho^3} - \lambda_t\,. \end{aligned} \tag{108}$$

The function F is calculated using the electromagnetic potentials and eqs. (28–30). The result is

$$\begin{aligned} F &= \frac{1}{2}(\rho\dot{\boldsymbol{\alpha}} - \dot{\rho}\boldsymbol{\alpha})^2 + \dot{\rho}(\rho\dot{\boldsymbol{\alpha}} - \dot{\rho}\boldsymbol{\alpha})\cdot\mathbf{q} + \rho(\rho\ddot{\boldsymbol{\alpha}} - \ddot{\rho}\boldsymbol{\alpha})\cdot\mathbf{q} + \\ &+ \frac{1}{2}(\dot{\rho}^2 + \rho\ddot{\rho})\mathbf{q}^2 + \frac{1}{\rho}(\rho\dot{\boldsymbol{\alpha}} - \dot{\rho}\boldsymbol{\alpha})\cdot\boldsymbol{\Omega}\times\mathbf{q} + G\,\lambda\,. \end{aligned} \tag{109}$$

Notice the presence of the gauge function λ in the last term of the equation (109).

All the ingredients to construct the Noether invariant (12) are already obtained. We arrive at

$$I = \frac{1}{2}\left(\rho(\dot{\mathbf{q}} - \dot{\boldsymbol{\alpha}}) - \dot{\rho}(\mathbf{q} - \boldsymbol{\alpha}) - \frac{\boldsymbol{\Omega}\times\mathbf{q}}{\rho}\right)^2 + \bar{V}(\bar{x},\bar{y})\,. \tag{110}$$

Remarks:

a) For $\Omega = 0$, the invariant (110) recovers the two–dimensional version of an invariant derived by Lewis [6], in his search for quadratic invariants for three–dimensional non–relativistic charged particle motion. For $\Omega \neq 0$, that is, when we include also rotations, our result is new.

b) Interestingly, despite the dependence of the electromagnetic potentials and the function F on the gauge function λ, the resulting invariant (110) is gauge independent. This is just what we expected a priori, since the choice of gauge should have no influence on physical quantities.

To provide an interpretation of the invariant, we make a change of variables. Using canonical group coordinates $\bar{\mathbf{q}} = (\bar{x}, \bar{y}, 0)$ and $\bar{t}$, the Lagrangian function reads

$$L = \frac{1}{\rho^2}\bar{L}(\bar{\mathbf{q}}', \bar{\mathbf{q}}) + \frac{dW}{dt}\,, \tag{111}$$

where

$$\bar{L}(\bar{\mathbf{q}}', \bar{\mathbf{q}}) = \frac{1}{2}\bar{\mathbf{q}}'^2 + \bar{\mathbf{A}}(\bar{\mathbf{q}}) \cdot \bar{\mathbf{q}}' - \bar{V}(\bar{\mathbf{q}})\,, \tag{112}$$

$$\begin{aligned} W &= \lambda + \frac{\dot{\rho}\mathbf{q}^2}{2\rho} + (\rho\dot{\boldsymbol{\alpha}} - \dot{\rho}\boldsymbol{\alpha}) \cdot \frac{\mathbf{q}}{\rho} + \\ &- \int^t \frac{d\mu}{\rho^2(\mu)}(\rho(\mu)\dot{\boldsymbol{\alpha}}(\mu) - \dot{\rho}(\mu)\boldsymbol{\alpha}(\mu))^2\,. \end{aligned} \tag{113}$$

The primes denote differentiation with respect to $\bar{t}$. The function W defined by eq. (113) can be disregarded in the Lagrangian, as it only adds a total derivative. Moreover, using $\bar{t}$ as the new time parameter, the action functional (2) reads

$$S = \int \bar{L}(\bar{\mathbf{q}}', \bar{\mathbf{q}})d\bar{t}\,, \tag{114}$$

from which it is evident that $\bar{L}$ may be used as the Lagrangian for the motion described in terms of canonical group coordinates. Indeed, using $\bar{L}$, the Lorentz equations become

$$\bar{x}'' = -\bar{V}_{\bar{x}}(\bar{x}, \bar{y}) + \bar{y}'\,\bar{B}(\bar{x}, \bar{y})\,, \tag{115}$$

$$\bar{y}'' = -\bar{V}_{\bar{y}}(\bar{x}, \bar{y}) - \bar{x}'\,\bar{B}(\bar{x}, \bar{y})\,, \tag{116}$$

which are the equations for two–dimensional non–relativistic charged particle motion in a time–independent electromagnetic field. We observe that the relation

$$\bar{\mathbf{q}}' = \frac{d\bar{\mathbf{q}}}{d\bar{t}} = \rho\, R^{-1}(T) \cdot \left(\dot{\mathbf{q}} - \frac{\boldsymbol{\eta}}{\rho^2}\right) \tag{117}$$

is useful to obtain the transformed Lagrangian and Lorentz equations.

The transformed equations of motion, being autonomous, have an associated energy-like invariant, of the form

$$I = \frac{1}{2}\bar{\mathbf{q}}'^2 + \bar{V}(\bar{x},\bar{y})\,, \tag{118}$$

which is precisely the Noether invariant (110) in transformed coordinates. Thus, the Noether constant of motion is the energy expressed in the variables where the equations of motion are autonomous.

5.2 The case $\rho = 0$ and $\Omega \neq 0$

The vector potential is, in this case,

$$\mathbf{A} = (y - \beta_2, -(x - \beta_1), 0)\frac{\psi(\bar{x},\bar{y})}{\bar{x}^2} + \nabla\lambda\,, \tag{119}$$

where $\lambda = \lambda(x, y, t)$ is the gauge function. The scalar potential V is now given by

$$\begin{aligned} V &= -\ddot{\beta}_1(x - \beta_1) - \ddot{\beta}_2(y - \beta_2) + \bar{V}(\bar{x},\bar{y}) + \\ &+ \frac{1}{\bar{x}^2}\left(\dot{\beta}_1(y - \beta_2) - \dot{\beta}_2(x - \beta_1)\right)\psi(\bar{x},\bar{y}) - \lambda_t\,, \end{aligned} \tag{120}$$

where

$$\bar{V}_{\bar{x}} = -\bar{x}\bar{E}_1(\bar{x},\bar{y})\,. \tag{121}$$

The resulting function F is

$$F = \dot{\beta}_2(x - \beta_1) - \dot{\beta}_1(y - \beta_2) + G\lambda\,, \tag{122}$$

and the associated Noether invariant is

$$I = (y - \beta_2)(\dot{x} - \dot{\beta}_1) - (\dot{x} - \beta_1)(y - \beta_2) + \psi(\bar{x}, \bar{y})\,. \tag{123}$$

To interpret Noether's invariant, again we change variables. Let the new coordinates of configuration space be $(\bar{x}, \bar{t})$. Using these variables, the Lagrangian becomes

$$L = \bar{L} + \frac{dW}{d\bar{y}}\,, \tag{124}$$

where

$$\bar{L} = \frac{1}{2}(\dot{\bar{x}}^2 + \bar{x}^2\dot{\bar{t}}^2) - \dot{\bar{t}}\psi(\bar{x}, \bar{y}) - \bar{V}(\bar{x}, \bar{y}) \tag{125}$$

is a new Lagrangian and

$$W = \lambda + (\dot{\beta}_1 \cos\bar{t} + \dot{\beta}_2 \sin\bar{t})\bar{x} + \frac{1}{2}\int^t \left(\dot{\beta}_1^2(\mu) + \dot{\beta}_2^2(\mu)\right) d\mu\,. \tag{126}$$

In this new description, $(\bar{x}, \bar{t})$ are the dependent variables and $\bar{y}$ is the independent variable.

In equation (124), it is apparent that the Lagrangian in the new coordinates can be taken as simply $\bar{L}$, since the addition of a total time derivative does not influence the equations of motion. Since $\bar{t}$ is a cyclic coordinate the momentum conjugate to $\bar{t}$,

$$p_{\bar{t}} = \bar{L}_{\dot{\bar{t}}} = \bar{x}^2\dot{\bar{t}} - \psi(\bar{x}, \bar{y})\,, \tag{127}$$

is a conserved quantity. This conserved quantity (127) is, apart from an irrelevant sign, the Noether invariant (123), which we can be interpreted as the conserved momentum conjugated to the cyclic coordinate $\bar{t}$.

5.3 The case $\rho = 0$, $\Omega = 0$ and $a_2 \neq 0$

Now, the vector potential is

$$\mathbf{A} = (0, \psi, 0) + \nabla\lambda \,. \tag{128}$$

where $\lambda = \lambda(x, y, t)$ is the gauge function, while the scalar potential is

$$V = -\frac{\ddot{a}_1}{a_2}xy + \frac{1}{2a_2^2}(a_1\ddot{a}_1 - a_2\ddot{a}_2)\, y^2 + \bar{V}(\bar{x}, \bar{y}) + \frac{\dot{a}_2 y}{a_2}\psi(\bar{x}, \bar{y}) - \lambda_t \,. \tag{129}$$

Using these electromagnetic potentials, we arrive at the function

$$F = \dot{a}_1\, x + \dot{a}_2\, y + G\lambda \,, \tag{130}$$

so that the Noether invariant is

$$I = -(a_1\dot{x} + a_2\dot{y} - \dot{a}_1 x - \dot{a}_2 y + a_2\psi(\bar{x}, \bar{y})) \,. \tag{131}$$

Again Noether's invariant is a gauge independent quantity.

For the interpretation of the invariant, we use $\bar{x}$ and $\bar{t}$ as new dependent variables and $\bar{y}$ as new independent variable. The Lagrangian can be expressed as

$$L = \bar{L} + \frac{dW}{d\bar{y}} \,, \tag{132}$$

where now

$$\bar{L} = \frac{1}{2}(\dot{\bar{x}}^2 + (a_1^2 + a_2^2)\dot{\bar{t}}^2) + (a_1\dot{\bar{x}} - \dot{a}_1\bar{x})\dot{\bar{t}} + a_2\psi(\bar{x}, \bar{y})\dot{\bar{t}} - \bar{V}(\bar{x}, \bar{y}) \tag{133}$$

is a new Lagrangian function, and,

$$W = \lambda + \dot{a}_1\bar{x}\bar{t} + \frac{1}{2}(a_1\dot{a}_1 + a_2\dot{a}_2)\bar{t}^2 \,. \tag{134}$$

The function W can be disregarded in the Lagrangian, as it enters $\bar{L}$ merely in the form of a total time derivative. For the Lagrangian $\bar{L}$, $\bar{t}$ is a cyclic variable, yielding the conserved momentum

$$p_{\bar{t}} = \bar{L}_{\dot{\bar{t}}} = (a_1^2 + a_2^2)\dot{\bar{t}} + a_1\dot{\bar{x}} - \dot{a}_1\bar{x} + a_2\psi(\bar{x}, \bar{y})\,, \tag{135}$$

which, apart from an irrelevant sign, is the Noether invariant (131). In conclusion, the Noether invariant may be interpreted as the momentum conjugated to the cyclic coordinate $\bar{t}$.

6 Conclusion

We have found the class of electromagnetic fields compatible with Noether symmetries for Lagrangians of type (1), describing physically interesting non–relativistic charged particle motion. The treatment comprises the complete resolution of the basic system of partial differential equations (34–36) by use of canonical group coordinates. There are three classes of electromagnetic fields yielding action functionals endowed with Noether invariance, as listed in section 4. These electromagnetic fields are consistent with Maxwell equations and depend on several arbitrary functions. The corresponding Noether invariants were explicitly shown in section 5, one in the form of an energy-like function (110) and two in the form of momentum-like functions, (123) and (131).

For a possible extension of the present work, we mention the investigation of the fully three–dimensional case. While trivial in principle, this extension may present, in practice, difficult mathematical problems. As a further development, one can analyze the complete integrability of the Lorentz equation corresponding to the electromagnetic fields associated with Noether point symmetries. From Liouville's theorem, two constants of motion are sufficient for the complete integrability of the equations of motion of a Hamiltonian system with two degrees of freedom, as in the present case. The explicit dependence on time does not modify this statement [11]. In the present work, we have derived classes of electromagnetic fields yielding just one constant of motion. For complete integrability, there is the need for a second invariant. This quantity exist only for special forms, found within the classes of electromagnetic fields constructed here. Consequently, the complete integrability of these systems remains an open question. As a final remark, the Noether invariants derived here may be useful in the construction of exact time–dependent solutions for the self-consistent Vlasov-Maxwell system in plasma physics.

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