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On the Lie symmetries of a class of generalized Ermakov systems

J. Goedert, F. Haas

Instituto de Física, UFRGS, Caixa Postal 15051, 91500-970 Porto Alegre, RS, Brazil

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Abstract

The symmetry analysis of Ermakov systems is extended to the generalized case where the frequency depends on the dynamical variables besides time. In this extended framework, a whole class of nonlinearly coupled oscillators are viewed as a Hamiltonian Ermakov system and exactly solved in closed form. © 1998 Elsevier Science B.V.

1. Introduction

The first study of an Ermakov system was published by the end of the last century [1], but the motivation for a detailed investigation of Ermakov systems intrinsic properties, their proper characterization and their application scope derived mainly from the works of Lewis who rediscovered them in the sixties [2], and of Lewis and Riesenfeld, who devised one of their most important applications [3], the quantization of the time-dependent harmonic oscillator.

In its original formulation, the Ermakov system was nothing but a time-dependent harmonic oscillator coupled to Pinney's [4] equation. Later on, Ray and Reid [5] extended the basic concept by incorporating two extra arbitrary functions in the original equations. Such a generalization highly improved the reach of the Ermakov systems with no harm to their key property, namely the existence of a constant of motion, the Ermakov-Lewis invariant.

Early developments dealt with Ermakov systems where the frequency [5] depended solely on time. In a subsequent generalization [6], the dynamical variables were allowed to participate in the argument of the frequency. The resulting generalized systems have rather interesting properties and potential applications [7,8], but has for the most remained a fertile field for further prospects. In particular, the Lie point symmetry analysis of Ermakov systems performed by Leach et al. [9-12] considered frequency functions depending at most on time. Inspired by the results of Leach et al., we reconsider the issue of geometric symmetries for *generalized* Ermakov systems, where the frequency is a function of both the dynamical variables and their first derivatives besides time. By doing this we find a much wider class of dynamical systems that aside from possessing the Ermakov-Lewis invariant admit a Lie symmetry in a direct and natural generalization of that proposed by Leach. As a first useful application of this extended viewpoint we show that some systems which would otherwise not fit in the older framework are readily treated as Ermakov systems.

This Letter is organized as follows. In Section 2, the class of generalized Ermakov systems with Lie point symmetry is determined. In Section 3, a class of

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coupled nonlinear oscillators are shown to constitute a generalized Ermakov system that possesses geometric symmetries and are exactly solvable in closed form. The conclusions are presented in Section 4.

2. The symmetry analysis

All Ermakov systems, both usual and generalized, can be represented [8] by the pair of equations

$$\ddot{x} + \Omega^2 x = \frac{1}{yx^2} F(y/x), \tag{1}$$

$$\ddot{y} + \Omega^2 y = 0, \tag{2}$$

where F is an arbitrary function of the indicated argument and Ω, hereafter called the generalized frequency, is an arbitrary function of the time, the coordinates and their derivatives. For physical reasons, we restrict the study to the cases where the frequency depends at most on the velocities, that is, we take $\Omega = \Omega(t, x, y, \dot{x}, \dot{y})$ only.

The pair of equations (1) and (2) encompasses the whole class of Ermakov systems in two spatial dimensions. In the traditional notation, the Ermakov systems were written in terms of *three* arbitrary functions [5] and not only *two* as we have proposed [8] elsewhere. This more concise representation is possible thanks to the generalized character of Ω. The usual representation is obtained whenever Ω is chosen in the form

$$\Omega^2 = \omega^2(t) - \frac{1}{xy^3} g(x/y), \tag{3}$$

where $g(x/y)$ and ω are arbitrary functions of the indicated arguments.

The use of generalized frequencies considerably expands the reach of the Ermakov system concept. Among the systems that can be treated as particular Ermakov systems in the generalized form, we quote the Kepler–Ermakov system [13] and the Lutzky's integrable system [14].

In order to perform the Lie symmetry analysis of generalized Ermakov system (1) and (2) we follow the approach of Govinder and Leach for the usual case [9,12], and initially consider the symmetries of the equivalent system

$$x\ddot{y} - y\ddot{x} + \frac{1}{x^2} F(y/x) = 0, \tag{4}$$

$$\ddot{y} + \Omega^2(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) y = 0, \tag{5}$$

where the notation $\boldsymbol{x} = (x, y)$ was introduced. The advantage of dealing first with the equivalent equation (4) derives from its independence of Ω. This feature provides a straightforward way of finding its symmetry generator, which is basically that obtained by Govinder and Leach [12] in the symmetry analysis of the usual Ermakov systems,

$$G = \rho^2 \frac{\partial}{\partial t} + \rho\dot{\rho}\boldsymbol{x} \cdot \frac{\partial}{\partial \boldsymbol{x}}, \tag{6}$$

where $\rho(t)$ is an arbitrary differentiable function. As pointed out in Ref. [12], the Ermakov first integral I is invariant under the first extension of G.

For the complete Ermakov system, that is, for equations (4) and (5) to obey the same symmetry, some restriction must be imposed on the functional form of Ω. This restriction is determined by requiring the invariance of Eq. (5) under the action of the first extension of G. We apply this criterion, and obtain the most general frequency that preserves the invariance of the generalized Ermakov system under the action of G,

$$\Omega^2 = -\frac{\ddot{\rho}}{\rho} + \frac{1}{\rho^4} \sigma(\boldsymbol{x}/\rho, \rho\dot{\boldsymbol{x}} - \dot{\rho}\boldsymbol{x}), \tag{7}$$

where σ is an arbitrary function of the indicated arguments. Notice that a velocity dependence is also possible, a fact that considerably expands the class of Ermakov systems invariant under the symmetry transformations of G.

To summarize the results so far we note that with no a priori assumption on the functional form of the frequency, we find that the most general Ermakov system possessing Lie point symmetry must be of the form

$$\ddot{x} + \left(\frac{\sigma}{\rho^4} - \frac{\ddot{\rho}}{\rho}\right) x = \frac{1}{yx^2} F(y/x), \tag{8}$$

$$\ddot{y} + \left(\frac{\sigma}{\rho^4} - \frac{\ddot{\rho}}{\rho}\right) y = 0. \tag{9}$$

where σ is arbitrary, but restricted to

$$\sigma = \sigma(\boldsymbol{x}/\rho, \rho\dot{\boldsymbol{x}} - \dot{\rho}\boldsymbol{x}). \tag{10}$$

The associated transformation group is that generated by G.

Let us show now that the whole class of usual Ermakov systems are recovered as special cases of the present treatment. For this purpose we use the relation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{\Omega_0^2}{\rho^3}, \tag{11}$$

to *define* a new function $\omega(t)$ in terms of ρ and the constant, but otherwise arbitrary, Ω_0. Despite its appearance of a differential equation, Eq. (11) can be alternatively considered as the definition of ω in terms of ρ. Let us adopt this viewpoint and also define a transformed $\bar{\sigma}$

$$\bar{\sigma}(x/\rho, \rho\dot{x} - \dot{\rho}x) = -\Omega_0^2 + \frac{\rho^4}{xy^3}g(x/y) + \sigma, \tag{12}$$

where we introduced, for convenience, a spurious but otherwise innocuous function $g(x/y)$, which can be absorbed in the definition of σ. These redefinitions transform the generalized form of the Ermakov system (8) and (9) into its more traditional form [5]

$$\ddot{x} + \left(\omega^2(t) + \frac{\bar{\sigma}}{\rho^4}\right)x = \frac{1}{yx^2}f(y/x), \tag{13}$$

$$\ddot{y} + \left(\omega^2(t) + \frac{\bar{\sigma}}{\rho^4}\right)y = \frac{1}{xy^2}g(x/y). \tag{14}$$

In Eqs. (13) and (14) we replaced

$$F(y/x) = f(y/x) + \frac{x^2}{y^2}g(x/y) \tag{15}$$

in order to facilitate the comparison with previous results on the Lie group of usual Ermakov systems. We should stress, however, that the superfluous function $f(y/x)$ was introduced only in order to reconstruct the traditional form.

We now notice that the pair of equations (13) and (14) reduces to the usual form with simple time dependence in the frequency only when $\bar{\sigma} \equiv 0$. For this particular class of frequencies the quasi-invariance transformation [15]

$$\bar{x} = x/C, \quad \bar{y} = y/C, \quad \bar{t} = \int \mathrm{d}t/C^2, \tag{16}$$

removes ω [16] from the equations of motion, which become

$$\bar{x}'' = \frac{1}{\bar{y}\bar{x}^2}f(\bar{y}/\bar{x}), \quad \bar{y}'' = \frac{1}{\bar{x}\bar{y}^2}g(\bar{x}/\bar{y}), \tag{17}$$

whenever $C(t)$ is a solution of the time-dependent harmonic oscillator

$$\ddot{C} + \omega^2(t)C = 0. \tag{18}$$

The symmetry group of the transformed Ermakov system (17) is $SL(2,R)$ as already shown in Ref. [9]. This, however, is not the case with Eqs. (13) and (14), which are more general than the Ermakov system with $SL(2,R)$ symmetry. The extra symmetry freedom of the system (13) and (14) derives from $\bar{\sigma}$, which can depend on the coordinates or the velocities. Notice, however, that the transformations belonging to the $SL(2,R)$ group are among the transformations generated by G.

The symmetries considered so far do not require any particular form for the function σ, nor depend on any preliminary coordinate transformation like Eq. (16). In this respect, it should be noted that, despite its apparent simplicity, the transformation (16) is not well defined when ω depends on the dynamical variables, a situation in which Eq. (18) becomes meaningless. In addition, even for $\omega = \omega(t)$, the time-dependent harmonic oscillator (18) does not always possess a closed form solution, a fact that may preclude its application in many interesting practical situations. Similar remarks apply to the linearization of the Ermakov system proposed in Ref. [16], which starts exactly with the transformation (16). Needless to say, such a linearization process applies only to the simplified system (17) and not to the generalized Ermakov system (8) and (9). However, it should be stressed that other generalized Ermakov systems involving extensions to multicomponents and higher dimensions have recently been introduced, which do have the underlying linear structure of the Ray–Reid system [17,18].

3. Hamiltonian generalized Ermakov systems

The Ermakov systems with $SL(2,R)$ symmetry in two spatial dimensions are solvable in closed form [11]. The more general Ermakov systems (8) and (9), however, are not always exactly solvable and this, certainly, is the price paid for the extra generality. Some additional structure, perhaps a Hamiltonian formalism, must exist in order to re-obtain equations that allow for exact solutions. To present one application example and to gain insight in this respect, let us consider the two-dimensional oscillator with nonlinear coupling introduced by Ray and Reid [19],

$$\ddot{x} + \omega^2(t)x = x\rho^{-4}G(xy/\rho^2), \tag{19}$$

$$\ddot{y} + \omega^2(t)y = y\rho^{-4}G(xy/\rho^2), \tag{20}$$

where $\rho(t)$ is an auxiliary function satisfying Pinney's equation,

$$\ddot{\rho} + \omega^2(t)\rho = 1/\rho^3. \tag{21}$$

Ray and Reid found a first integral for Eqs. (19) and (20), namely

$$J = (\rho\dot{x} - \dot{\rho}x)(\rho\dot{y} - \dot{\rho}y) + \frac{xy}{\rho^2} - \int^{xy/\rho^2} G(\tau)\,\mathrm{d}\tau, \tag{22}$$

but did not obtain its complete solution. In this section we show that the generalized Ermakov system approach combined with the symmetry analysis, yields the corresponding complete solution, in closed form.

Let us first show that Eqs. (19) and (20) are in fact a generalized Ermakov system. This is easily verified by choosing in Eqs. (8) and (9), $F = 0$ and Ω a function of the coordinates

$$\Omega^2 = \omega^2(t) - \frac{1}{\rho^4}G(xy/\rho^2). \tag{23}$$

From this viewpoint the amplitude of the angular momentum

$$I = \tfrac{1}{2}(x\dot{y} - y\dot{x})^2, \tag{24}$$

becomes the corresponding Ermakov–Lewis invariant. Moreover, Eqs. (19) and (20) form a Hamiltonian Ermakov system, a property that has already played a major role in several applications [8,20,21]. For the present case the Hamiltonian is

$$H = p_x p_y + \omega^2 xy - \frac{1}{\rho^2}\int^{xy/\rho^2} G(\tau)\,\mathrm{d}\tau, \tag{25}$$

which is not a constant of motion, since the system is non-conservative.

The frequency (23) is of the right form (7) and therefore guarantees the invariance of Eqs. (19)–(20). The use of the auxiliary equation (21) to substitute ρ for $\omega(t)$ transforms Ω into

$$\Omega^2 = -\frac{\ddot{\rho}}{\rho} + \frac{1}{\rho^4}(1 - G(xy/\rho^2)), \tag{26}$$

which is indeed of the general form (7).

The Hamiltonian structure and the symmetry can now be further exploited to reduce the problem to quadrature. For this purpose, the use of canonical group coordinates,

$$u = x/\rho, \quad v = y/\rho, \quad T = \int \rho^{-2}\,\mathrm{d}t, \tag{27}$$

is an essential tool that transforms the symmetry generator into a simple time translation, $G = \partial/\partial T$. In these new variables, the equations of motion become the autonomous system

$$u'' + u = uG(uv), \qquad v'' + v = vG(uv), \tag{28}$$

where primes denote derivatives with respect to the new time. Moreover, this system is described by the Hamiltonian

$$K = p_u p_v + uv - \int^{uv} G(\tau)\,\mathrm{d}\tau, \tag{29}$$

which is the transformed version of the first integral J given by Eq. (22). The Hamiltonian K in the canonical group coordinates is quadratic in momentum. In such cases [8], the appropriated variables for quadrature are

$$q = uv, \quad s = v/u. \tag{30}$$

In these coordinates, the constants of motion become

$$\sqrt{2I} = qs'/s, \tag{31}$$

$$J = (q'^2 - 2I)/4q + q - \int^{q} G(\tau)\,\mathrm{d}\tau. \tag{32}$$

The quadrature of the last two equations successively give $q(T)$ and $s(T)$. The map back to the original variables is the pure algebraic task of using

$$x^2 = \rho^2 q/s, \quad y^2 = \rho^2 qs, \tag{33}$$

to substitute (x, y) for (q, s). One final step is the determination of $T(t)$ which can be obtained by the integration of the last equation in Eq. (27). In the quadrature sense, this completes the integration process. Recall that all the necessary structure and the correct choice of variables was dictated by the Hamiltonian character, by the generalized Ermakov form of the system and by the presence of symmetry. In particular, the analytical form of the solution can be found explicitly in terms of elliptic functions when $G(\tau) = c_1/\tau^2 + c_2 + c_3\tau + c_4\tau^2$, where c_i are arbitrary constants.

4. Conclusion

Generalized Ermakov systems (8) and (9) are of a very general nature, and their complete application scope is still not fully outlined. In this Letter we determined their Lie point symmetry group and showed that a set of nonlinear coupled oscillators belonging to this class of systems is in fact explicitly soluble in term of quadratures. This provides a stimulating preview of the variety of applications in which generalized Ermakov systems may play an important role.

To conclude, we mention that Lie symmetries of the type generated by Eq. (6) and some modifications of it occur in several problems in physics. Some non-Hamiltonian systems with velocity-dependent frequency, a whole class of two- and three-dimensional charged particle motions, some perturbed magnetic monopole and a time-dependent Kepler problem fall in the category of such systems [22] and are frequently amenable to exact solution in closed form.

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