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On the generalized Hamiltonian structure of 3D dynamical systems

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Abstract

The Poisson structures for 3D systems possessing one constant of motion can always be constructed from the solution of a linear PDE. When two constants of motion are available the problem reduces to a quadrature and the structure functions include an arbitrary function of them.

1. Introduction

Three-dimensional dynamical systems have received much attention both in view of their intrinsic mathematical relevance and of their wide interest in domains such as mechanics [1], optics [2,3], dynamics of interacting populations [4-7], modeling of fluid turbulence [8,9], wave interaction models [10-12], dynamo theory [13], and several other areas of physical, chemical or biological importance. A more recent related issue concerns the Poisson structures of these systems. This question has been contemplated both from the point of view of their existence [14] and of their explicit determination [15-18]. As a rule, 3D systems possess a Poisson structure whenever a sufficient number (e.g. *two*) of independent constants of motion exist [19]. Their explicit construction, however, is a partly open problem which, as we show in this Letter, can always be solved when two constants of motion are known.

In a recent paper Gümral and Nutku [16] reduced the problem of finding the Poisson structure of 3D dynamical systems to the solution of a quasi-linear partial differential equation (Eq. (70) of their paper) in three independent variables. When two independent constants of motion are known, the problem was therefore reduced to a Ricatti equation. In this Letter we show that in fact the Poisson structures of 3D systems possessing one constant of motion can always be obtained from the solutions of a *linear* partial differential equation. When *two* independent constants of motion are known the problem is reduced to a quadrature. In such cases the resulting structure functions involve an arbitrary function of the constants of motion. A Poisson structure involving such arbitrary functions occurred earlier in the study of the symmetric top [15] and was recently shown to be a general feature of bi-Hamiltonian systems in three dimensions [16].

When only one constant of motion is available, the problem can frequently be handled in the sense that a generalized Poisson formalism may still be constructed in terms of some particular solution of the pertaining equations. To illustrate the procedure we shall consider examples of both types. An interesting system with only one known constant of motion is a five-parameter version of the 3D Lotka-Volterra system. As we show in Section 4.2, a Hamiltonian formalism can be constructed for this system in terms of a partic-

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ular solution of the basic equation (13) in our treatment. As far as we could check, no other independent constant of motion is known for the five-parameter Lotka–Volterra system and its complete integrability is therefore an open question. A four-parameter version of the same system is known to possess two independent constants of motion and, consequently, to admit a bi-Hamiltonian structure [6].

2. Generalized Hamiltonian structures

In this section we consider a generic dynamical system in N dimensions,

$$\dot{x}^\mu = v^\mu(\boldsymbol{x}, t), \quad \mu = 1, \ldots, N, \tag{1}$$

where v^μ is a sufficiently smooth vector field (in general $v^\mu \in \mathbb{C}^\infty$), $\boldsymbol{x} = (x^1, x^2, \ldots, x^N)$, and the dot denotes the derivative with respect to t. In addition we consider a function $H(\boldsymbol{x}, t)$ satisfying

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}, \tag{2}$$

along any phase trajectory, that is

$$v^\mu \partial_\mu H = 0, \tag{3}$$

where ∂_μ indicates the partial derivative with respect to x^μ and repeated indices represent the Einstein summation convention. In specific applications the function H is typically a time-independent first integral of (1) valid over a region in phase space. Finally we consider an *antisymmetric* matrix $\mathcal{J}$ of elements $J^{\mu\nu}$ which satisfies the Jacobi identities [20],

$$J^{\mu[\alpha} \partial_\mu J^{\beta\gamma]} = 0, \tag{4}$$

and therefore provides a generalized definition for the Poisson bracket,

$$[F, G] \equiv \partial_\mu F J^{\,mu\nu} \partial_\nu G, \tag{5}$$

of functions F and G in phase space.

Definition. System (1) is said to be Hamiltonian *iff* there exists a function H satisfying (2) and an antisymmetric matrix $\mathcal{J}$ satisfying the Jacobi identities such that

$$v^\mu \equiv J^{\mu\nu} \partial_\nu H. \tag{6}$$

In such a case, H is called the Hamiltonian of (1) and $\mathcal{J}$ the associated Poisson tensor or matrix of structure functions.

Let v^μ be the vector field in (1) and $H(\boldsymbol{x}, t)$ be its Hamiltonian, that is, a solution of (3). Notice that the dynamical variables can always be labeled such that $\partial_N H \neq 0$. Let us now consider an arbitrary antisymmetric matrix of elements $J^{\mu\nu}$. Under these conditions, the equality

$$v^\mu = J^{\mu\nu} \partial_\nu H, \quad \text{for } \mu = 1, \ldots, N-1 \tag{7}$$

implies

$$v^N = J^{N\nu} \partial_\nu H. \tag{8}$$

This simple statement, which can be easily verified, has a strong implication: if a dynamical system possesses *one* constant of motion and $N-1$ of its equations are cast in a Hamiltonian-like form, then the remaining equation automatically acquires the same Hamiltonian form. The proof of the proposition follows from the antisymmetry of $J^{\mu\nu}$ and from the fact that H is a nontrivial constant of motion. The importance of the proposition can be rephrased as follows: when (7) holds, $N-1$ components of the original antisymmetric tensor $J^{\mu\nu}$ can be expressed in terms of the remaining $\frac{1}{2}(N-1)(N-2)$ components,

$$J^{\mu N} = -J^{N\mu} = \left(v^\mu - \sum_{\nu=1}^{N-1} J^{\mu\nu} \partial_\nu H \right) \Big/ \partial_N H. \tag{9}$$

This result can be obtained by solving (7) for $J^{N\nu}$.

Thus, any function satisfying (3) recasts the system (1) in the "pre-Hamiltonian" form (6). The algorithm consists of taking $\frac{1}{2}(N-1)(N-2)$ arbitrary functions $J^{\mu\nu}$ ($\mu < \nu = 1, \ldots, N-1$) and filling up the $M \times N$ antisymmetric matrix with them, their antisymmetric counterpart and the $J^{N\mu}$ given by (9). The hamiltonization process will be completed by demanding that $J^{\mu\nu}$ be a Poisson tensor, that is, obeys the Jacobi identities.

To close this section we remark that in general the Jacobi identities form an overdetermined system of nonlinear equations in the unknown $J^{\mu\nu}$. In fact, when a constant of motion exists they form a set of $N!/3!(N-3)!$ equations in $\frac{1}{2}(N-1)(N-2)$ unknowns $\{J^{\mu\nu} : \mu < \nu = 1, \ldots, N-1\}$. For $N < 3$ the Jacobi identities are automatically satisfied and the hamiltonization process is trivial. For $N > 3$ the resulting system of equations is usually overdetermined. For $N = 3$ (which is the object of the following sections) they become a *linear* partial differential equation. Finally we remark that the statement involving Eqs. (7), (8) can be used to explore the existence of a Poisson structure of dynamical systems of dimension higher than three. In four dimensions, for example, there are *four* Jacobi identities and only *three* unknown components $J^{\mu\nu}$. Thus, when the system under consideration involves an additional arbitrary function the problem will be well posed in terms of the three unknown components of the Poisson tensor and the extra arbitrary function. Some applications on this line are being prepared and the results will be submitted for publication elsewhere. In this Letter, however, we shall restrict our attention to three-dimensional systems.

3. Three-dimensional systems

For $N = 3$, Eqs. (7) read

$$v^1 = J^{12}\partial_2 H + J^{13}\partial_3 H, \tag{10}$$

$$v^2 = -J^{12}\partial_1 H + J^{23}\partial_3 H. \tag{11}$$

Let us solve Eqs. (10), (11) in terms of H, the vector field v^μ and some function J to be determined later,

$$J^{12} = J, \quad J^{13} = \frac{v^1 - J\partial_2 H}{\partial_3 H},$$

$$J^{23} = \frac{v^2 + J\partial_1 H}{\partial_3 H}. \tag{12}$$

Recall that a different labeling of the variables must be adopted when $\partial_3 H \equiv 0$. In such cases Eqs. (12) must undergo a cyclic permutation of the indices ($1 \to 2 \to 3 \to 1$) in order to exchange ∂_3 with either ∂_2 or ∂_1. This permutation must be applied simultaneously to Eqs. (14), (15) below.

Eqs. (12) express all the components of the Poisson tensor in 3D in terms of the symbol J to be determined by the Jacobi identity. To see this we substitute (12) into (4) and multiply the result by $\partial_3 H$. The cross derivative and the quadratic terms cancel out, and, after using (3) and some simple algebra, we obtain

$$v^\mu \partial_\mu J = AJ + B, \tag{13}$$

where

$$A = \partial_\mu v^\mu - \frac{\partial_3 v^\mu \partial_\mu H}{\partial_3 H} \tag{14}$$

and

$$B = \frac{v^1\partial_3 v^2 - v^2\partial_3 v^1}{\partial_3 H}. \tag{15}$$

Eq. (13), which is the Jacobi identity in terms of J, is the key equation in the solution of the hamiltonization problem of 3D systems. As mentioned before, this last condition is a linear first order PDE. When two constants of motion are known, one of the characteristic equations for (13) becomes a linear first order ODE (two out of the three dynamical variables are solved in terms of the third one and the constants of motion). In this case J can be solved by quadratures and the structure functions may always involve an arbitrary function of the two constants of motion. As already mentioned, this feature is not novel (Hojman [15] constructed a Poisson structure for the symmetric top in terms of arbitrary functions of the two independent constants of motion in the problem and, more recently, Gümral and Nutku [16] presented a formalism in which this possibility was indicated for 3D systems in general). The advantage of the formalism presented here is the linear character of the fundamental equation (13). This linear form favors the existence of general solutions in case of complete integrability. This is the case, for example, in the ice skate problem studied in Section 4.1 below.

It is important to remark that the scale invariance of the Poisson structure of 3D systems [16] is preserved by Eqs. (13)–(15): if H is a Hamiltonian and C is a Casimir with associated Poisson tensor (12) (expressed in terms of J) then $F(H,C)$, for arbitrary F, is another possible Hamiltonian but with associated Poisson tensor $\bar{J} \equiv J/(\partial F/\partial H)$. In fact, as it can be checked [21], if J satisfies Eq. (13) for A and B calculated using H, then $\bar{J}$ satisfies the same equation with A and B calculated using $F(H,C)$ as the new Hamiltonian. The multiplicity of solutions for the hamiltonization process of 3D systems is therefore twofold: it depends on the overall scale (or conformal) invariance and on the presence of an arbitrary (in general additive) function that characterizes the general solutions of the first order partial differential Eq. (13). This latter possibility will be shown explicitly in Section 4, for the Poisson structure of the ice skate problem.

As a final remark of practical interest we point that any particular solution to (13) provides a nontrivial Poisson structure for the system under consideration. Interesting particular cases are obtained when the system has $B = 0$, in which case $J = 0$ is the simplest possible solution or when $B(\boldsymbol{x}) = -f(H,C)A(\boldsymbol{x})$ in which case $J = f(H,C)$ is an equally simple particular solution. The last example presented in Section 4.2 explores a similar possibility for the Lotka–Volterra system with five free parameters for which only one constant of motion is known in the literature [4].

4. Sample applications

To illustrate some of the various possibilities offered by the procedure proposed above we consider now two sample problems in detail. Several other systems can be equally treated by repeating a similar sequence of steps. A comprehensive and detailed list of application examples has recently been presented [22] for currently studied 3D systems which admit *rescalable* time-dependent constants of motion.

4.1. The ice skate problem

As a first example we consider the ice skate problem studied by Lucey [17]. In the original treatment one of the equations was $\dot{x}^2 = 0$, which can be solved for $x^2 = -a =$ const. This transforms the 4D system into a 3D one. We next relabel the variables using the replacements $x^4 \to x^3 \to x^2$. In this notation the ice skate system reads

$$\dot{x}^1 = -a, \quad \dot{x}^2 = x^3, \quad \dot{x}^3 = ax^3 \tan x^1. \tag{16}$$

We now verify that $H_1 = x^3 \sec x^1$ and $H_2 = ax^2 + x^3 \tan x^1$ are two independent constants of motion and therefore good candidates as the Hamiltonian for the system. Using H_1 as a source in (14), (15) gives $A = 0$ and $B = -a \cos x^1$. We next verify that one of the characteristic equations of (13),

$$\frac{\mathrm{d}J}{\mathrm{d}x^1} = \cos x^1, \tag{17}$$

is separated from the others and can be integrated in the form

$$J = \sin x^1 + F(H_1, H_2), \tag{18}$$

where F is an arbitrary function of the constants of motion. The following structure functions are now obtained by substitution of (18) in (12)

$$\begin{aligned} J^{12} &= -J^{21} = \sin x^1 + F(H_1, H_2), \\ J^{13} &= -J^{31} = -a \cos x^1, \\ J^{23} &= -J^{32} = x^3[\sec x^1 + F(H_1, H_2) \tan x^1]. \end{aligned} \tag{19}$$

The Casimir of the associated algebra is given by the solution of

$$\frac{\partial C}{\partial H_1} - F(H_1, H_2)\frac{\partial C}{\partial H_2} = 0, \tag{20}$$

and depends on the choice of the arbitrary function $F(H_1, H_2)$. In particular the choice $F = 0$, implies $C = H_2$. Eq. (20) can be alternatively regarded as the relation to determine F. From this viewpoint one can take for the Casimir any arbitrary function of H_1 and H_2 and solve (20) for F. A similar possibility applies for the alternative Poisson structure of the ice skate system (Eq. (23) below).

The ice skate problem is in fact completely integrable since H_2 is another constant of motion, functionally independent of H_1. We may therefore use H_2 as an alternative Hamiltonian in which case $A = B = -a \cot x^1$ and

$$\bar{J} = -1 + \bar{F}(H_1, H_2) \sin x^1, \tag{21}$$

for any arbitrary function $\bar{F}$ of the constants of motion. The resemblance between expressions (18) and (21) (and consequently between the related pairs (19), (22) and (20), (23)) suggests a more concise representation, which unfortunately is not possible. We proceed with the calculation and obtain

$$\begin{aligned} \bar{J}^{12} &= -\bar{J}^{21} = -1 + \bar{F}(H_1, H_2) \sin x^1, \\ \bar{J}^{13} &= -\bar{J}^{31} = -a\bar{F}(H_1, H_2) \cos x^1, \\ \bar{J}^{23} &= -\bar{J}^{32} = x^3[-\tan x^1 + \bar{F}(H_1, H_2) \sec x^1]. \end{aligned} \tag{22}$$

The Casimir of the new associated algebra is now given by the solution of

$$\frac{\partial C}{\partial H_2} + \bar{F}(H_1, H_2)\frac{\partial C}{\partial H_1} = 0. \tag{23}$$

For $\bar{F} = 0$ the new Casimir is $C_2 = H_1$. It is worthwhile to remark that the algebras corresponding to J and $\bar{J}$ are independent and not connected by conformal transformations.

We have presented a complete and novel solution to the hamiltonization of the ice skate system. This problem illustrates clearly the basic features of the routine proposed. In the next subsection we analyze another sample problem and show that a well-known system may still exhibit some novel and perhaps unexpected features.

4.2. The 3D Lotka–Volterra system

The 3D Lotka–Volterra system and some of its special subsystems play an important role in modeling many physical, chemical and biological processes. Their associated vector field is defined (in the most general form with Verhulst terms $b_{ii} \neq 0$) by

$$v^k \equiv x^k(a_k + b_{k\mu}x^\mu), \quad k = 1, \ldots N. \tag{24}$$

In Eq. (24) and throughout the rest of this Letter summation over k is not implied.

Cairó and Feix [4] found several invariants or first integrals for N-dimensional Lotka–Volterra systems. In particular for $N = 3$ and $\det(b_{ij}) = 0$, one such invariant is

$$\mathcal{H} = H(x^1, x^2, x^3)\, e^{-st} \equiv (x^1)^\alpha (x^2)^\beta (x^3)^\gamma\, e^{-st}, \tag{25}$$

where α, β, γ and s are given by

$$\alpha = b_{22}b_{31} - b_{21}b_{32}, \quad \beta = b_{11}b_{32} - b_{12}b_{31},$$
$$\gamma = b_{12}b_{21} - b_{11}b_{22}, \quad s = a_1\alpha + a_2\beta + a_3\gamma.$$

Under the constraint $s = 0$, that is, for

$$a_1(b_{22}b_{31} - b_{21}b_{32}) + a_2(b_{11}b_{32} - b_{12}b_{31}) + a_3(b_{12}b_{21} - b_{11}b_{22}) = 0, \tag{26}$$

the function H becomes a time-independent first integral. Below we present the derivation of a Poisson structure for this system under the less restrictive conditions that we can find. This will ultimately imply that among the initially free coefficients a_k and b_{ij} only five remain arbitrary. Unfortunately no other constant of motion is known for the same number of free parameters and complete integrability cannot be sought. It is, however, interesting to verify that under appropriate conditions one can construct a Poisson structure without being actually able to integrate the equations completely.

In order to tackle the basic equation (13) we first calculate A and B in terms of the vector field v^μ and of the constant of motion H. For this, Eqs. (14), (15) and (25) yield

$$A = a_1 + a_2 + b_{11}x^1 + b_{22}x^2 + b_{33}x^3 + (b_{1\mu} + b_{2\mu})x^\mu, \tag{27}$$

$$B = \frac{U}{\gamma H}[(a_1b_{23} - a_2b_{13}) + (b_{23}b_{1\mu} - b_{13}b_{2\mu})x^\mu], \tag{28}$$

where we introduced $U = x^1x^2x^3$ to simplify the notation. To proceed we write the characteristic equation associated to (13)

$$\frac{dx^k}{x^k(a_k + b_{k\mu}x^\mu)} = \frac{dJ}{AJ + B}, \tag{29}$$

which after some tedious but elementary algebra implies

$$\frac{dU}{U[A + a_3 + (b_{31} - b_{11})x^1 + (b_{32} - b_{22})x^2]} = \frac{dJ}{AJ + B}. \tag{30}$$

We now multiply the numerator and the denominator of the first term in Eq. (30) by $\epsilon/\gamma H$, where ϵ is an arbitrary constant to be determined later, and rewrite the characteristic equations in the form

$$\frac{dx^k}{x^k(a_k + b_{k\mu}x^\mu)} = \frac{d(J - \epsilon U/\gamma H)}{A(J - \epsilon U/\gamma H) - B'(U/\gamma H)}, \tag{31}$$

with B' defined by

$$B' = a_1b_{23} - a_2b_{13} - \epsilon a_3 + [\epsilon(b_{11} - b_{31}) + b_{23}b_{11} - b_{13}b_{21}x^1 + [\epsilon(b_{22} - b_{32}) + b_{23}b_{12} - b_{13}b_{22}]x^2. \tag{32}$$

Eq. (31) is difficult to treat in its general form. One particular solution, however, can be readily found by imposing the additional condition $B' = 0$. As can be easily checked, under this condition

$$J = \frac{\epsilon U}{\gamma H} = \frac{\epsilon}{\gamma}(x^1)^{1-\alpha}(x^2)^{1-\beta}(x^3)^{1-\gamma} \tag{33}$$

satisfies Eq. (31) and therefore the fundamental equation (13). The condition on B' produces the value of the arbitrary constant ϵ and (for arbitrary x^1 and x^2) two additional constraints on the coefficients of the system. This represents a total of *four* constraints ($\det(b_{ij}) = 0$, $s = 0$ and two of the conditions (34)–(36) below) on the *twelve* initially free parameters. This corresponds to a total of eight free parameters, which in fact reduces to *five* since three parameters can always be eliminated by rescaling.

Condition (32) for arbitrary x^1 and x^2, implies three equations of which one can be solved for ϵ when either (i) $a_3 \neq 0$ or (ii) $b_{31} \neq b_{11}$ or (iii) $b_{32} \neq b_{22}$. These conditions imply the following relations,

$$(a_1 b_{23} - a_2 b_{13})(b_{31} - b_{11}) = a_3(b_{23}b_{11} - b_{13}b_{21}), \tag{34}$$

$$(a_1 b_{23} - a_2 b_{13})(b_{32} - b_{22}) = a_3(b_{23}b_{12} - b_{13}b_{22}), \tag{35}$$

$$(b_{23}b_{11} - b_{13}b_{21})(b_{32} - b_{22}) = (b_{23}b_{12} - b_{13}b_{22})(b_{31} - b_{11}), \tag{36}$$

which we interpret as follows:

(i) when $a_3 \neq 0$ apply (34), (35) and use

$$\epsilon = \frac{1}{a_3}(a_1 b_{23} - a_2 b_{13}); \tag{37}$$

and/or (ii) when $b_{31} \neq b_{11}$ apply (34), (36) and use

$$\epsilon = \frac{b_{23}b_{11} - b_{13}b_{21}}{b_{31} - b_{11}}; \tag{38}$$

and finally (iii) when $b_{32} \neq b_{22}$ apply (35), (36) and calculate ϵ from

$$\epsilon = \frac{b_{23}b_{12} - b_{13}b_{22}}{b_{32} - b_{22}}. \tag{39}$$

To complete the calculation we insert ϵ in (33) and substitute J into (12). This provides a Poisson structure for the 3D Lotka–Volterra system, namely

$$J^{12} = -J^{21} = \frac{\epsilon}{\gamma H} x^1 x^2 x^3, \tag{40}$$

$$J^{13} = -J^{31} = \frac{x^1 x^3}{\gamma H}\left(b_{1\mu}x^\mu - \frac{\epsilon\beta}{\gamma}x^3\right), \tag{41}$$

$$J^{23} = -J^{32} = \frac{x^2 x^3}{\gamma H}\left(b_{2\mu}x^\mu + \frac{\epsilon\alpha}{\gamma}x^3\right). \tag{42}$$

In Nutku's analysis [6] of the Lotka–Volterra system, one of the first integrals was the logarithm of the function used above and no Verhulst (diagonal) terms were included. Like in the present treatment the determinant of the coefficient of the quadratic terms was set to zero and an additional constraint was imposed similar to condition (26). This slightly more constrained system (four free parameters) has a known second constant of motion and consequently admits a bi-Hamiltonian structure. The component J^{12} of the Poisson tensor in Nutku's analysis satisfies condition (13) but is a different particular solution.

5. Conclusions

We have considered a procedure for constructing the Poisson structure of dynamical systems possessing a constant of motion. In three dimensions the problem is reduced to the solution of a linear PDE. When two time-independent first integrals of the system are known the procedure can be applied twice and yields bi-Hamiltonian structures involving *arbitrary* functions of the constants of motion. The technique was applied to a few sample systems to show how it works. In particular we verified that the ice skate system analyzed by Lucey [17] is completely integrable and that a five free parameter version of the 3D Lotka–Volterra system can be cast into a generalized Hamiltonian form. Other 3D systems for which one or more time-independent constants of motion are known can be treated equally.

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References

[1] V. Arnold, Mathematical methods of classical mechanics (Springer, Berlin, 1989).
[2] D.D. Holm and B. Wolf, Physica D 51 (1991) 189.
[3] M. Puta, C. R. Acad. Sci. 318 (1994) 679.
[4] L. Cairó and M.R. Feix, J . Math. Phys. 33 (1992) 2440.
[5] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, J.-M. Strelcyn and S. Wojciechowski, Physica A 163 (1990) 683.
[6] Y. Nutku, Phys. Lett. A 145 (1990) 27.
[7] Y. Nutku, J. Phys. A 23 (1990) L1145.
[8] E.N. Lorenz, J. Atmos. Sci. 20 (1963) 130.
[9] M. Kuś, J. Phys. A 16 (1983) L689.
[10] H.J. Giacomini, C.E. Repetto and O.P. Zandron, J. Phys. A 24 (1991) 4567.
[11] T.C. Bountis, A. Ramani, B. Grammaticos and B. Dorizzi, Physica A 128 (1984) 268.
[12] A.S. Pikovski and M.I. Rabinovich, Rev. Math. Phys. 2 (1981) 165.
[13] T. Rikitake, Proc. Cambridge Phys. Soc. 54 (1957) 89.
[14] V. Perlick, J. Math. Phys. 33 (1992) 599.
[15] S.A. Hojman, J. Phys. A 24 (1991) L249.
[16] H. Gümral and Y. Nutku J. Math. Phys. 34 (1993) 5691.
[17] C.A. Lucey, J. Math. Phys. 29 (1988) 2430.
[18] F. Haas and J. Goedert, Hamiltonian structure for rescaled integrable Lorenz systems, in: Proc. II Encontro Brasileiro de Física dos Plasmas, Serra Negra, SP, October 1993, pp. 118–121.
[19] M. Razavy and F.J. Kennedy, Can. J. Phys. 52 (1974) 1532.
[20] P.J. Olver, Graduate texts in mathematics, Vol. 107. Applications of Lie groups to differential equations (Springer, Berlin, 1986).
[21] F. Haas, Formulações de Poisson para sistemas dinâmicos, Msc. Thesis, Instituto de Física da UFRGS, Porto Alegre, August 1994.
[22] J. Goedert, F. Haas, D. Hua, M.R. Feix and L. Cairó, J. Phys. A 27 (1994) 6495.