Environmental policy and directed innovation in a Schumpeterian growth model.

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ABSTRACT: This paper describes the main long-run properties of an endogenous growth model with carbon emissions. Growth comes from innovation, which is produced by an R and D sector. Innovations can be either energy-augmenting or labor-augmenting. We consider the impact of a carbon tax as well as a subsidy to energy-augmenting R & D. While the former has moderate effects in inducing cleaner technologies in the long run, the latter is much more efficient. However, only an exploding carbon tax may implement a 'sustainable' growth path where output grows while emissions fall.

JEL: Q25, Q28, O3
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1. Introduction

This paper describes an endogenous growth model of the world economy with carbon emission. This model serves as the basis for a calibrated empirical model being developed at IDEI in Toulouse. In this paper we describe the building blocks of the model as well as the main properties of its steady state.

An influential model of climate change and growth has been developed by William Nordhaus of Yale University (Nordhaus, 1994; Nordhaus and Boyer, 2000). It predicts, in particular, that a carbon tax achieves very little in reducing emissions, while having substantial costs in terms of GDP. As a result, the optimal tax is minute, implying that arrangements such as the Kyoto protocol are harmful from the point of view of social welfare.

It has long been recognized (see for example Carraro and Siniscalco (1994)) that the standard growth models may be inappropriate in assessing the long-run costs and benefits of environmental policy, because they ignore how technology itself may change in response to relative prices[1]. However, there have only been few attempts to properly take care of these effects in a fully specified growth model.

This is what the model presented here undertakes. I assume the existence of induced innovative bias in response to carbon taxes. Productivity growth is the outcome of R and D activity which introduces new intermediate goods in the fashion of Romer (1990) and Grossman and Helpman (1991). This R and D activity can be directed ex-ante toward inventing goods which reduce the energy input, or goods which reduce the labor input. A carbon tax affects the relative profitability of the first type of innovation compared to the second one. If parameters are such that it goes up, then an increase in carbon taxes triggers a reallocation of R and D labor toward the energy-saving sector. The energy content of output gradually goes down as innovations of this sort are introduced, until a new steady state is reached.

[1] This endogenous technology response seems substantial, as shown by Popp (2001), who looks at the effects of energy prices on patents.

In the existing literature, the paper which is to my knowledge most related to the present one is Bovenberg and Smulders (1995,1996). It shares two features with the present model, namely that technical change is endogenous and increases the efficiency of the pollution-intensive input. However, it differs in two main respects. First, all innovation is energy-augmenting, whereas in the present model it can be directed to another sector. Second, there are no intellectual property rights: knowledge appears as a pure externality as in the AK models of Romer (1986) and Rebelo (1990), and private investment in knowledge accumulation only takes place if subsidized by the government. A similar "AK" approach is used by Byrne (1997), who, contrary to Bovenberg and Smulders, assume knowledge to be output-augmenting. There, people invest in their own technology and it is the only accumulable factor, which solves the problem of incentives for knowledge accumulation. In contrast, in the present paper innovation takes the form of new intermediate goods on which the inventor holds monopoly power as in Romer (1990). This is also true in Aghion and Howitt (1998, chapter 5); they analyze pollution and limited natural resources in the context of 'Schumpeterian' endogenous growth models, and show that sustainable positive growth exists in the long-run if there is enough substitutability between natural resources (or the pollution-intensive input) and knowledge. However, as there is only one R & D sector, they do not discuss the role of induced innovation, nor how it responds to environmental policies.

In the rest of the paper, I lay out the model's assumption, discuss how one can compute the steady state, analyze numerical simulations of the steady state. I then look at the effects of a constant carbon tax, a subsidy to 'green' R &D, and an exploding carbon tax. I finally discuss the implications of a version of the model with an endogenous number of R & D workers.

2. Factor demand and pricing

The aggregate production function is given by:

$$Y_t = f(X_{1t}, X_{2t}) = \left[b_1 X_{1t}^{\alpha} + b_2 X_{2t}^{\alpha}\right]^{1/\alpha},$$

where Y_t is total output, X_{1t} an aggregate of intermediate goods using energy as their input, and X_{2t} an aggregate of intermediate goods using labor as their input.[2]

Following Romer (1990), each aggregate is defined as

$$X_{it} = \left[\int_0^{n_{it}} z_{it}(\omega)^{\varepsilon_i} d\omega \right]^{1/\varepsilon_i},$$

where $z_{it}(\omega)$ is the amount of intermediate good intensive in energy (resp. labor) for $i = 1$ (resp. $i = 2$). Thus, there is a continuum of mass $n_{1t}+n_{2t}$ of intermediate goods, each being indexed by (i, ω).

Each intermediate good is produced under a constant returns to scale production function, where the corresponding input is the only factor, and where input requirements are normalized to unity.[3] Total energy consumption is thus equal to

$$E_t = \int_0^{n_{1t}} z_{1t}(\omega) d\omega$$

Employment in the physical goods sector is given by

$$L_t = \int_0^{n_{2t}} z_{2t}(\omega) d\omega$$

As in the literature on endogenous growth, each intermediate good is owned by a monopoly, which charges monopoly prices. The derived demand for intermediate good (i, ω) is, if its price is $p_{it}(\omega)$:

$$z_{it}(\omega) = k_i p_{it}(\omega)^{-\sigma_i},$$

where

$$k_i = X_{it} \left(\int_0^{n_{it}} p_{it}(\omega)^{-\frac{\varepsilon_i}{1-\varepsilon_i}} d\omega \right)^{-1/\varepsilon_i},$$

and $\sigma_i = 1/(1 - \varepsilon_i)$ is the price elasticity of demand for good i.

[2] Capital does not appear, but this is not a very important issue, as the model is to be used for very long run analysis, and capital adjusts quite quickly.

[3] This assumption is just a normalization. There is no loss of generality since measurement units can always be set so that it is true.

Given that each producer of an intermediate good is infinitesimal, it considers k_i as given and therefore applies monopoly pricing under a constant demand elasticity:

$$p_{it}\left(\omega\right)=\frac{\sigma_i}{\sigma_i-1}q_{it},$$

where q_{it} is the price of input i. We will denote by $q_t = q_{1t}$ the price of energy (inclusive of the energy tax) and wages by $w_t = q_{2t}$. The output of intermediate good (i,ω) and the corresponding monopoly profit is given by

$$z_{it}\left(\omega\right)=X_{it}n_{it}^{-\sigma_i/(\sigma_i-1)}=z_{it},\forall\omega$$

$$\pi_{it}(\omega)=X_{it}n_{it}^{-\sigma_i/(\sigma_i-1)}q_{it}/(\sigma_i-1)=\pi_{it}.$$

By symmetry, these quantities do not depend on ω. Total cost of intermediate good i to the producer of the final good is therefore given by $n_{it}z_{it}p_{it}$, i.e.:

$$n_{it}^{-1/(\sigma_i-1)}\left(\frac{\sigma_i}{\sigma_i-1}\right)q_{it}X_{it}=q_{it}X_{it}/a_{it}\left(\frac{\sigma_i}{\sigma_i-1}\right),$$

where $a_i = n_i^{1/(\sigma_i-1)}$. In this formula, a_{it} is an aggregate productivity index for composite good i, therefore $X_{1t} = E_t a_{1t}$ and $X_{2t} = L_t a_{2t}$. The RHS tells us that the total input cost of composite good i is equal to the total input requirement of factor i, X_{it}/a_{it}, times the cost of factor i, q_{it}, times the markup $\frac{\sigma_i}{\sigma_i-1}$. An important aspect of that formula is that the introduction of new intermediate goods is equivalent, at the aggregate level, to an increase in the productivity of the corresponding input. Romer's economic interpretation of that property, is that an increase in the number of intermediate inputs is equivalent to a greater division of labor in production, i.e. each intermediate input is interpreted as a "task" and there are gains from dividing production into a greater number of narrower tasks. While our analysis will use a_i as state variable, the number of goods can be recovered using:

$$n_i = a_i^{\sigma_i-1}$$

The final good sector is produced by competitive firms, implying that the demand for composite good i is given by

$$X_{it} = Y_t \left[\frac{b_i a_{it}}{q_{it}} \frac{\sigma_i - 1}{\sigma_i} \right]^{1/(1-\alpha)}.$$

Finally, we impose that the price index of the final good is normalized to one, i.e.

$$1 = b_1^{1/(1-\alpha)} \left(\frac{q_t}{a_{1t}} \frac{\sigma_1}{\sigma_1 - 1} \right)^{-\alpha/(1-\alpha)} + b_i^{1/(1-\alpha)} \left(\frac{w_t}{a_{2t}} \frac{\sigma_1}{\sigma_1 - 1} \right)^{-\alpha/(1-\alpha)}.$$

This normalization implies that all nominal quantities are expressed in terms of the final good.

3. Emissions and Energy taxes

In this model, all energy is fossil. Carbon emissions are proportional to energy consumption, and parameter c_0 reflects the physical carbon content of one unit of energy. [4]

$$CO_2 = c_0.E_t.$$

Thus, one simplification of this model is that the CO_2/E ratio remains constant. Reductions in emissions can only be obtained by three channels: (i) a reduction in output, (ii) substitution of the labor-intensive composite good for the energy-intensive one (but labor supply will be assumed to be inelastic), (iii) productivity growth in a_1 which reduces the energy requirement for one unit of the energy-intensive composite good. However, the economic impact of technical

[4] More generally, one could consider several distinct greenhouse gases, several primary inputs with different emission contents, and several composite inputs. If a_{ijt} is the requirement for aggregate composite good i in primary input j at date t, and ϕ_{jk} the unit emission of gas k by input j, total emissions of gas k are given by

$$EM(k) = \sum_i X_{it} \sum_j a_{ijt} \phi_{jk},$$

or; in vector form:

$$EM = (A_t \Phi)' X_t)$$

progress in the carbon content of energy can still be analyzed within the limits of this simplified model, by looking at the effects of an exogenous decline in c_0. More fundamentally, an extension of the model where c_0 is endogenous because a fraction of intermediate goods use non-fossil energy is currently being studied.

The model can be used to consider a variety of fiscal instruments. The two most relevant ones are a carbon tax, which, given our assumptions, is equivalent to an energy tax, and a subsidy to energy-oriented R and D. We assume that the energy tax is paid by producers of intermediate goods using energy. Therefore, at any point in time, the price of energy q_t is related to the production price of energy $\bar{q}_t$ by

$$q_t = \bar{q}_t(1 + \tau_t),$$

where τ_t is the tax. Tax revenues are rebated to consumers in a lump-sum fashion and therefore do not appear in their optimality conditions.

4. Innovation

Innovation is made by people who work in the R and D sector, or "researchers". They invent new intermediate goods in either sector. They represent a fraction θ of total population N. While this number is fixed, there is free mobility of researchers between R and D devoted to inventing new intermediate goods in sector 1 (energy-intensive) and R and D devoted to inventing new intermediate goods in sector 2 (labor-intensive).

Inventing a good gives an infinitely lived patent on production of that good. The present discounted value of such a monopoly right evolves according to

$$r_t V_{it} = \dot{V}_{it} + \pi_{it}$$

The cost of inventing a new good in terms of R and D workers is given by c_i/n_{it}. As in Grossman and Helpman, the externality in n_{it} guarantees that innovation generates sustained growth at a strictly positive rate. Otherwise, the rate at which new goods are invented $\dot{n}_{it}$ would eventually fall relative to the existing stock of goods n_{it}, and growth would tend to zero.

We allow for the government to subsidize R and D in sector i at rate s_i. In practice we will only consider subsidies to "green" R and D, i.e. $s_2 = 0$. As the total number of researchers is fixed, this goes without loss of generality; only the relative returns between R and D in sector 1 and R and D in sector 2 matter. The subsidy is financed by a lump-sum tax on consumers, and therefore does not distort their optimality conditions.

The R and D sector is competitive and firms in that sector take the wage of researchers as given. Consequently, in equilibrium, this wage w_{Rt} must satisfy:

$$\max_{i=1,2} \left(V_{it} - c_i(1 - s_i) w_{Rt}/n_{it} \right) = 0$$

This condition tells us that the net marginal present value of the most profitable R and D firms must be zero, i.e. that the asset value of an extra patent is exactly equal to its development cost $c_i w_{Rt}(1 - s_i)/n_{it}$.

There are 3 possible regimes.

1. If $\frac{n_1 V_1}{c_1(1-s_1)} > \frac{n_2 V_2}{c_2(1-s_2)}$, then all researchers innovate in sector 1.

2. If $\frac{n_1 V_1}{c_1(1-s_1)} = \frac{n_2 V_2}{c_2(1-s_2)}$, they are indifferent between the two sectors and endogenously spread themselves in both.

3. If $\frac{n_1 V_1}{c_1(1-s_1)} < \frac{n_2 V_2}{c_2(1-s_2)}$, they all innovate in sector 2.

Intuitively, which regime prevails in the long run should depend on the elasticity of substitution in production between the energy-intensive and the labor-intensive aggregate. If it is large, then an increase in productivity in sector 1 should increase the demand and the price of energy-intensive intermediate inputs, so that innovation in that sector becomes more profitable. One should then expect all researchers to be specialized in one sector; i.e. the economy will be in regime 1 or 3, depending on initial conditions. If elasticity of substitution is low, then innovation in one sector creates a bottleneck in the other sector, thus increasing the value of innovating in that sector, and the economy should converge to regime 2.

5. Labor markets

In the physical goods sector, labor supply is inelastic. A fraction $(1-\theta)$ of total population supplies one unit of labor to that sector. Equilibrium in the labor market implies that this supply must be equal to demand at any date:

$$L_t = (1-\theta)N = a_{2t}X_{2t}$$

In the R and D sector, innovation of both types must use the total number of researchers. Calling μ_t the fraction of researchers devoted to sector 1, this implies:

$$\dot{a}_1 = \frac{\mu_t N\theta a_{1t}}{c_1(\sigma_1 - 1)} \tag{5.1}$$

$$\dot{a}_2 = \frac{(1-\mu_t)N\theta a_{2t}}{c_2(\sigma_2 - 1)} \tag{5.2}$$

6. Consumption

There exists a representative consumer who maximizes

$$\int_0^{+\infty} C_t^{\gamma}/\gamma \cdot e^{-\rho t} dt.$$

Financial markets are perfect and there is no uncertainty. The Euler condition is therefore

$$\frac{\dot{C}_t}{C_t} = \sigma_C(r_t - \rho),$$

where $\sigma_C = 1/(1-\gamma)$ and r_t is the instantaneous real interest rate.

7. Extraction

In many models it is typically assumed that fossil energy is a natural resource in fixed initial supply, implying that a Hotelling-style intertemporal arbitrage condition must hold for the price of energy. In a full-fledged growth model, this assumption makes it difficult to obtain a balanced growth path, except under special assumptions[5]. Furthermore, it adds one forward-looking condition, which substantially increases the complexity of numerical simulations. Finally, assuming a fixed total supply of natural resources, known by all, is not necessarily realistic: new resources are regularly discovered, provided one spends enough on prospection and extraction costs—presumably these costs go up as one moves toward resources that are more costly to locate and extract. Our modelling of the extraction sector captures these properties, while escaping the difficulties of the Hotelling model. We assume that at each date t people can freely extract energy at a marginal cost which increases both with the total stock of energy extracted in the past and with the flow being currently extracted. Thus we assume that the extraction technology uses the final good as its input and that the unit extraction cost is equal to $\phi Z_t^{\delta} E_t^{\beta+1}$, where $Z_t = \int_{-\infty}^{t} E_t dt$ is the total stock extracted in the past and $\beta, \delta > 0$. At date t, the marginal extraction cost is equal to $\phi Z_t^{\delta} E_t^{\beta}$, and we assume it is equal to the producer's price $\bar{q}_t$.

$$\bar{q}_t = (1 + \beta)\phi Z_t^{\delta} E_t^{\beta}$$

Note that producers do not internalize the effect of their current extraction activity on future extraction costs, via the increment in future values of Z_t. This will be true if there are no property rights on land. Otherwise $\bar{q}_t$ would obey a Hotelling-style intertemporal arbitrage equation, since the market value of land would reflect future marginal extraction costs.

8. Goods market equilibrium

Finally, the goods market equilibrium condition states that total output is allocated between consumption and extraction costs:

$$Y_t = C_t + \phi Z_t^{\delta} E_t^{\beta} E_t$$

[5] One example is Aghion and Howitt (1998, ch. 5).

9. Summary

The following table summarizes the equations of the model.

There are 18 endogenous variables: $X_1, X_2, a_1, a_2, E, Y, C, Z, \bar{q}, r, q, w, \pi_1, \pi_2, V_1$, V_2, μ, w_R, and 18 relationships among these variables (Equations (9.16)-(9.18) only define one relationship between μ, V_1, V_2, a_1, a_2, and w_R). Of these, there are three state variables: the technological levels a_1 and a_2 and the stock of past extractions Z. Furthermore, there are two forward looking variables whose expected rate of change affect intra-period equilibrium determination: V_1 and V_2.

Table 1 : Equations of the model

$$X_{2t} = (1-\theta)Na_{2t} \tag{9.1}$$

$$X_{1t} = a_{1t}E_t \tag{9.2}$$

$$Y_t = [b_1X_{1t}^{\alpha} + b_2X_{2t}^{\alpha}]^{1/\alpha} \tag{9.3}$$

$$Y_t = C_t + \phi Z_t^{\delta} E_t^{\beta} E_t \tag{9.4}$$

$$\dot{Z}_t = E_t \tag{9.5}$$

$$\bar{q}_t = (1+\beta)\phi Z_t^{\delta} E_t^{\beta} \tag{9.6}$$

$$\frac{\dot{C}_t}{C_t} = \sigma_C(r_t - \rho) \tag{9.7}$$

$$X_{1t} = Y_t \left[\frac{b_1 a_{1t}}{q_t}\frac{\sigma_1 - 1}{\sigma_1}\right]^{1/(1-\alpha)} \tag{9.8}$$

$$X_{2t} = Y_t \left[\frac{b_2 a_{2t}}{w_t}\frac{\sigma_2 - 1}{\sigma_2}\right]^{1/(1-\alpha)} \tag{9.9}$$

$$1 = b_1^{1/(1-\alpha)}\left(\frac{q_t}{a_{1t}}\frac{\sigma_1}{\sigma_1 - 1}\right)^{-\alpha/(1-\alpha)} + b_2^{1/(1-\alpha)}\left(\frac{w_t}{a_{2t}}\frac{\sigma_2}{\sigma_2 - 1}\right)^{-\alpha/(1-\alpha)} \tag{9.10}$$

$$q_t = \bar{q}_t(1+\tau_t) \tag{9.11}$$

$$\pi_{1t} = X_{1t}q_t a_{1t}^{-\sigma_1}(\sigma_1 - 1)^{-1} \tag{9.12}$$

$$r_t V_{1t} = \dot{V}_{1t} + \pi_{1t} \tag{9.13}$$

$$
\begin{aligned}
\pi_{2t} &= X_{2t} w_t a_{2t}^{-\sigma_2} (\sigma_2 - 1)^{-1} && (9.14)\\
r_t V_{2t} &= \dot{V}_{2t} + \pi_{2t} && (9.15)\\
0 &= \max_{i=1,2} \left(V_{1t} - c_1(1 - s_1) w_{Rt} a_{1t}^{1-\sigma_1}, V_{2t} - c_2(1 - s_2) w_{Rt} a_{2t}^{1-\sigma_2} \right) && (9.16)\\
0 &= \mu_t \left(V_{1t} - c_1(1 - s_1) w_{Rt} a_{1t}^{1-\sigma_1} \right) && (9.17)\\
0 &= (1 - \mu_t) \left(V_{2t} - c_2(1 - s_2) w_{Rt} a_{2t}^{1-\sigma_2} \right) && (9.18)\\
\dot{a}_1 &= \frac{\mu_t N \theta a_{1t}}{c_1(\sigma_1 - 1)} && (9.19)\\
\dot{a}_2 &= \frac{(1 - \mu_t) N \theta a_{2t}}{c_2(\sigma_2 - 1)} && (9.20)
\end{aligned}
$$

10. Balanced growth paths with a constant energy tax

This section describes how a balanced growth path (BGP) can be computed if the energy tax τ_t is a constant. The next section studies "semi-balanced" growth paths with an energy tax increasing at a constant rate.

10.1. Computing growth rates

Let us look for a solution to (9.1)-(9.20) such that any endogenous variable x grows at a constant rate g_x. Let us also look for an equilibrium in regime 2, so that $\left(V_{1t} - c_1(1 - s_1) w_{Rt} a_{1t}^{1-\sigma_1} \right) = \left(V_{2t} - c_2(1 - s_2) w_{Rt} a_{2t}^{1-\sigma_2} \right) = 0$. Because of the production function and the goods market equilibrium condition, consumption, output, and the aggregates X_1 and X_2 must grow at the same rate g. Furthermore, equations (9.1) and (9.2) imply that this common rate must be equal to that of a_2, g_{a_2}, also equal to $g_{a_1} + g_E$. Equation (9.10) implies $g_q = g_{\bar{q}} = g_{a_1}$ and $g_w = g_{a2}$. Equation (9.6) implies $\delta g_Z + \beta g_E = g_{\bar{q}} = g_{a_1}$. Given that in a BGP, one must have $g_E = g_Z$, it follows that $g_E = g_Z = g_{a_1}/(\delta + \beta)$. Using these relationships and eliminating μ_t between (9.19) and (9.20), we get the following condition, which allows to compute growth rates:

$$
g_{a_1} c_1(\sigma_1 - 1) + g_{a_2} c_2(\sigma_2 - 1) = \theta N
$$

Once this is done, all relevant growth rates can be computed. Equilibrium growth rates of all variables are summarized below:[6]

$$
\begin{aligned}
g_{a_1} &= g_q = g_{\bar{q}} = \frac{(\delta+\beta)\theta N}{c_1(\sigma_1-1)(\delta+\beta)+c_2(\sigma_2-1)(1+\delta+\beta)} && (10.1)\\
g_E &= g_Z = \frac{\theta N}{c_1(\sigma_1-1)(\delta+\beta)+c_2(\sigma_2-1)(1+\delta+\beta)} && (10.2)\\
g_{a_2} &= g_Y = g_C = g_w = \frac{\theta N(1+\delta+\beta)}{c_1(\sigma_1-1)(\delta+\beta)+c_2(\sigma_2-1)(1+\delta+\beta)} && (10.3)\\
g_{\pi_1} &= (2-\sigma_1)g_{a1} + g_E && (10.4)\\
g_{\pi_2} &= (2-\sigma_2)g_{a2} && (10.5)\\
g_r &= g_\mu = 0 && (10.6)
\end{aligned}
$$

Note that in this steady state, technical progress in the labor sector must be larger than in the energy sector. This is because the quantity of energy being extracted grows at a positive rate, while total employment does not. For growth in the aggregate energy-intensive input to match growth in the employment-intensive input, technical progress must be slower in the former sector. The reverse would occur if extraction were to grow at a negative rate. However, this is not the case in the steady states of this model. With a constant tax on energy, extraction must grow. One could try to construct a steady state where E grows at a negative rate, thus falling to zero, while Z converges to a constant level. In such a steady state, the marginal cost of energy must fall, and so must the price of energy. However, this is incompatible with the requirement that the price of energy should grow at the same rate as a_1, since there cannot be technical regress. Economically, this means that as energy gets cheaper and cheaper, the aggregate input X_1 will grow

[6] Note that for the economy to remain in regime 2 throughout, it must be that V_i grows at the same rate as $a_{it}^{1-\sigma_i}$, or equivalently:

$$g_{\pi_1} + (\sigma_1 - 1)\, g_{a_1} = g_{\pi_2} + (\sigma_2 - 1)\, g_{a_2}$$

One can check that this condition is indeed satisfied.

faster than output, which is inconsistent with being in a BGP, and would imply a positive growth rate of extraction, which is a contradiction.

This important result implies that a constant energy tax cannot achieve "sustainable development", if one defines it as a stabilization of carbon emissions. Further below, we consider what happens when instead of a constant tax rate there is a growing tax rate, and show that for a range of levels of the tax rate's growth rate, growth is "sustainable", i.e. output grows at a positive rate while extraction and emissions fall.

10.2. Computing levels

The preceding analysis computes the long-term growth rates that endogenous variables must have along a BGP. In the appendix, we show how the corresponding levels can be recursively computed. We assume an arbitrary initial value of a_1, a_{10}. We compute the corresponding initial values of all the other variables consistent with the economy being in a BGP from $t = 0$ onwards. That is, the set of initial values $\{x_0\}$ for any endogenous variable x, such that $x_t = x_0 e^{g_x t}$ is solution to (9.1)-(9.20), with g_x as determined in the previous subsection.

They are summarized by the following table:

Table 2. Determination of the levels

$$
\begin{aligned}
r &= \rho + g_{a_2}/\sigma \\
\mu &= \frac{g_{a_1} c_1 (\sigma_1 - 1)}{N\theta} \\
&\quad \frac{1}{\phi(1+\beta)(1+\tau_0)} a_{10}^{1+\delta+\beta} \lambda \left[a_{20}(1+\tau_0)^{1/(\delta+\beta)} a_{10}^{-(1+1/(\delta+\beta))} \right] a_{20}^{-(\delta+\beta)} \\
&= (N(1-\theta))^{\delta+\beta} \left(\frac{\sigma_1 c_1 (1-s_1)(r - g_{\pi_1})}{\sigma_2 c_2 (1-s_2)(r - g_{\pi_2})} \right)^{\frac{\delta+\beta}{\alpha}} g_E^{-\delta} \left(\frac{b_2}{b_1} \frac{\sigma_2 - 1}{\sigma_2} \frac{\sigma_1}{\sigma_1 - 1} \right)^{\frac{\delta+\beta}{\alpha}} \\
q_0 &= a_{10} \lambda \left[a_{20}(1+\tau_0)^{1/(\delta+\beta)} a_{10}^{-(1+1/(\delta+\beta))} \right] \\
w_0 &= \left[\frac{1}{\phi(1+\tau_0)(1+\beta)} \right]^{\frac{1-\alpha}{\delta+\beta}} q_0^{1+\frac{1-\alpha}{\delta+\beta}} g_E^{\frac{\delta(1-\alpha)}{\delta+\beta}} \left((1-\theta) N \right)^{-(1-\alpha)} \left(\frac{a_{20}}{a_{10}} \right)^{\alpha} \left(\frac{b_2}{b_1} \frac{\sigma_2 - 1}{\sigma_2} \frac{\sigma_1}{\sigma_1 - 1} \right)
\end{aligned}
$$

$$
\begin{aligned}
E_0 &= (1-\theta)N\left(\frac{a_{20}}{a_{10}}\right)^{-\frac{\alpha}{1-\alpha}}\left(\frac{q_0}{w_0}\frac{b_2}{b_1}\frac{\sigma_2-1}{\sigma_2}\frac{\sigma_1}{\sigma_1-1}\right)^{-1/(1-\alpha)} \\
X_{10} &= a_{10}E_0 \\
X_{20} &= a_{20}N(1-\theta) \\
Y_0 &= [b_1X_{10}^{\alpha}+b_2X_{20}^{\alpha}]^{1/\alpha} \\
C_0 &= Y_0-\phi E_0^{1+\delta+\beta}g_E^{-\delta} \\
\pi_{10} &= X_{10}q_0a_{10}^{-\sigma_1}(\sigma_1-1)^{-1} \\
\pi_{20} &= X_{20}w_0a_{20}^{-\sigma_2}(\sigma_2-1)^{-1} \\
V_{10} &= \frac{\pi_{10}}{r-g_{\pi_2}} \\
V_{20} &= \frac{\pi_{20}}{r-g_{\pi_2}} \\
\bar{q}_0 &= q_0/(1+\tau_0)
\end{aligned}
$$

In these formulas, the $\lambda(.)$ function is defined so that $\lambda(y)$ is the value of x solution to

$$
\kappa_0x^{-\alpha/(1-\alpha)}+\kappa_1x^{-\frac{\alpha}{1-\alpha}-\frac{\alpha}{\delta+\beta}}y^{\alpha}=1,
$$

where the coefficients κ_i are defined in the Appendix.

It is not clear whether this procedure gives a solution, but if it does, it is straightforward to check that the corresponding BGP is a solution to the model, i.e. satisfies (9.1)-(9.20). In practice, we have constructed a computer program which performs steps 1-6, and it does converge for a wide range of parameters. It can be used to numerically evaluate the properties of the balanced growth path.

10.3. Properties of the steady state

In this subsection we report numerical simulations results for the long-run steady state.[7] We first report a benchmark simulation with a set of parameters giving reasonable steady state properties. These parameters are reported in Table 3.

[7] The parameters used correspond to a rough, reasonable calibration and not to the actual set of parameters that will be used in the full model.

Table 3. Parameter Values	
α	-0.4
b_1	0.011
b_2	0.944
ϕ	1.51
τ_0	0.1
$s_1 = s_2$	0
σ_C	1
σ_1	2
σ_2	2
c_1	2.5
c_2	15
ρ	0.01
θ	0.02
N	20
β	0.35
δ	1.64

The value of σ_C is consistent with studies of the intertemporal elasticity of substitution in consumption, while σ_1 and σ_2 give plausible markups. The proportion of researchers is 2 %, consistent with observed values, while N was calibrated to get reasonable orders of magnitudes for the growth rate, i.e. around 2 to 2.5 %. The carbon tax is set at a benchmark level of 10 %, while we do not consider any subsidy to R and D. The rate of time preference is set at 1 %, which is at the low end, but appropriate for environmental problems. c_1 and c_2 were chosen to get an equilibrium share of researchers in the green sector of 10 %. α was taken equal to -0.4, in order to generate enough complementarity between labor and energy. b_1 and b_2 were calibrated so as to match the share of energy in value added and the wage level. The initial level of technology in sector 1, a_{10} was normalized to 1. δ and β were calibrated so as to match the ratio of the growth rate of oil prices over extraction in the long run (abstracting from oil shocks), and the estimated elasticity of extraction with respect to the stock already extracted.

The following Table gives the main endogenous variables of interest in the steady state.

Table 4. Endogenous variables in the benchmark simulation	
Variable	Value
$g = g_Y = g_{a_2} = g_C = g_w = g_{w_R}$	2.4 %
$g_{a_1} = g_q$	1.6 %
g_E	0.8 %
$r =$	3.4 %
$\mu =$	10 %
$a_{20} =$	1.1
$q_0 =$	19.07
$w_0 =$	0.42
$E_0 =$	0.0549
$X_{10} =$	0.0549
$X_{20} =$	21.69
$Y_0 =$	18.57
$C_0 =$	17.86
$\pi_{10} =$	1.05
$\pi_{20} =$	7.44
$V_{10} =$	40.36
$V_{20} =$	218.8
$Z_0 =$	6.85
$\bar{q}_0 =$	17.34
$w_{R0} =$	16.14
$s_E = qE/Y$	0.0564

This benchmark simulation implies a realistic growth rate of 2.4 %, and an energy share in GDP which is a bit high but not too offmark: 5.64 %–The actual share, is about 4 %–parameters were calibrated so as to match it under a zero energy tax rate; increasing that rate increases the energy share in expenditure. Note that the wages of researchers appear as much higher than production workers' wages; but this is essentially a matter of normalization: nothing would be changed if one were to double the number of researchers and halve their labor endowment (which is not commensurate with production work), or equivalently double b_1 and b_2. This would reduce the predicted wage of researchers by half.

10.4. Doubling Carbon taxes

We now run another simulation where the energy tax is twice as high, $\tau = 0.2$, and compare with the previous steady state. Note that one should be cautious with these comparisons, since levels are compared across points in time where $a_1 = 1$, rather than across identical dates. Hence for any variable x one cannot directly compare x_0 across the two steady states. Rather, one has to compute meaningful ratios.

We already know that the energy tax is not going to affect the long-run growth rates, nor the real interest rate. More surprisingly, it does not affect the long-run value of μ, the share of researchers devoted to green R and D. This is because there is only one value of μ consistent with the long-run growth rates, i.e. such that $g_{a_2} = g_{a_1} + g_E$, and it only depends on the parameters which determine these growth rates.

On the other hand, as Table 5 makes clear, the relative technical level in sector 2 falls by 3.7 %. Note that this should not be interpreted as an absolute fall in a_2, since in both simulations technological level in sector 1 is normalized to 1. What this means is that in the long-run technology is relatively more advanced in the energy-intensive sector than in the labor-intensive sector, by an amount equal to 3.7 %, if the energy tax is equal to 20 % rather than 10 %. Of course, if one starts from a steady state with $\tau = 10$ % and if one applies a once-and-for all doubling of the energy tax, in order to improve the relative technical level in sector 1 it must be that μ goes up. But such an increase cannot last forever as μ must eventually return to its long-run value. However this transitory event has long-lasting effects on the relative productivity in energy-intensive goods, which has gone up by 3.7 % permanently.

Given that a_1 is normalized to one, the value of q_0 should be interpreted as the (constant) value of q/a_1, which is nothing but the relative price of one unit of the energy-intensive aggregate X_1. Line 2 of Table 5 tells us that it is almost unchanged between the two steady states. This comes from two effects: on the one hand, energy is relatively more expensive, which increases the cost of X_1. On the other hand, technology is relatively more advanced in sector 1, which reduces the energy input requirement per unit of X_1. The two effects seem to balance each other almost exactly.

Line 3 of Table 5 reports one dimension of income distribution, namely the share of wages in final goods production (the rest goes to energy and to the profits of intermediate goods firms). As can be seen, this share is virtually constant. The next line looks at another dimension of income distribution, namely the ratio between researchers' wages and production workers wages. It is again virtually unchanged.

The next line looks at the fraction of output devoted to extraction, which is equal to $1-C/Y$, since all output is either consumed or dissipated in the extraction sector. This share falls from 3.8 % to 3.48 %.

The next ratio, $X_1/Y = a_1E/Y$, is a measure of the energy content of output. The proper measure, E/Y, falls at rate g, whereas this measure is constant along the steady state. This measure "asks more" from economies that are more productive in saving energy. It does not move for an economy whose energy use per unit of output falls exactly by the amount its productivity in using energy goes up. This is virtually what is going on here.

Table 5. Impact of a doubling of the carbon tax

Variable	Previous	New value	Rel. change
a_{20}	1.1	1.059	-3.7 %
q_0	19.07	19.07	0.0
$w_0N(1-\theta)/Y_0$	0.443	0.443	0.0
w_R/w_0	38.43	38.42	0.0
$1-C/Y$	3.8 %	3.48 %	-8.4 %
X_1/Y	0.00296	0.00296	0.0

10.5. A subsidy to "Green" R and D

These results can be compared to what one would obtain if instead one had a subsidy to energy-saving R and D. Note however that comparing carbon taxes with R and D subsidies is not obvious: which level of a subsidy is "comparable" to a given level of the tax? Here we take a not fully satisfactory stance, comparing the introduction of a 10 % subsidy to the 10 percentage point increase in energy taxes we have just considered. In both cases, at least, a relative price connected with the energy sector is distorted by 10 %. Thus we now report a simulation with $\tau_0 = 0.1$ and $s_1 = 0.1$.

A striking point is that such a policy is much more efficient in boosting the relative productivity of the energy-intensive sector than a carbon tax. This relative productivity now goes up by 34.5 %. The induced fall in energy demand is strong enough to generate a large fall in the relative price of the energy-intensive aggregate. As a result, more output goes to wages, while inequality between researchers and production workers falls very slightly. Finally, there is a slight fall in the extraction share, while adjusted energy intensity goes up, reflecting the large increase in the relative productivity of sector 1.

In short, this policy seems to have stronger effects on the structure of economic activity than the carbon tax. But only dynamic simulations will be able to tell us something about their cumulative impact on carbon emissions.

Table 6. Impact of a 10 % green R & D subsidy

Variable	Previous	New value	Rel. change
a_{20}	1.1	0.72	-34.5 %
q_0	19.07	13.72	-28 %
$w_0 N(1-\theta)/Y_0$	0.443	0.448	+1.12 %
w_R/w_0	38.43	38.42	0
$1 - C/Y$	3.8 %	3.46 %	-11.5 %
$b_1^{1/\alpha} X_1/Y$	0.00296	0.00374	+26.3 %

11. Increasing carbon taxes: the case for "sustainable development"

In the preceding simulations, carbon emissions and therefore the temperature grow without bounds. Therefore, there is a sense in which these balanced growth paths are not sustainable in the long run. This brings the following question: do there exist balanced growth paths that are sustainable, i.e. such that economic growth is positive while the temperature is stabilized in the long run? Aghion and Howitt (1998, ch.5) have provided an example of such a possibility, in the context of an endogenous growth model with a production function which is Cobb-Douglas in natural resources and knowledge capital. Grimaud (1999) has shown that the implementation of the social optimum in such a model involves an exponentially increasing time profile of carbon taxes. Therefore, we now study the determinants of long-run growth under an exponentially growing carbon tax, i.e. $(1+\tau_t) = (1+\tau_0)e^{g_\tau t}$.

The key difference relative to the preceding analysis is that emissions may now grow at a negative rate, meaning that the cumulative stock of CO2 in the atmosphere converges from below to a maximum (equal, up to a constant, to minus the long-run value of the stock of energy in the ground, Z). Therefore, one has to distinguish between two regimes:

Regime 1: $g_E \geq 0$.

In this case, we have $g_Z = g_E$ and denoting by $g = g_Y = g_{a_2}$ we get that

$$g_E = \frac{g - g_\tau}{1+\delta+\beta}$$

Next, we can compute g_{a_1} as a function of g and g_τ :

$$g_{a_1} = g - g_E = \frac{g(\delta+\beta)+g_\tau}{1+\delta+\beta};$$

and the growth rate can again be computed by writing down the equilibrium condition on the market for researchers:

$$\frac{(g(\delta+\beta)+g_\tau)c_1(\sigma_1-1)}{1+\delta+\beta} + gc_2(\sigma_2-1) = \theta N.$$

Thus,

$$g = \frac{\theta N(1+\delta+\beta) - g_\tau c_1(\sigma_1-1)}{c_1(\sigma_1-1)(\delta+\beta)+c_2(\sigma_2-1)(1+\delta+\beta)},$$

$$g_E = \frac{\theta N - g_\tau(c_1(\sigma_1-1)+c_2(\sigma_2-1))}{c_1(\sigma_1-1)(\delta+\beta)+c_2(\sigma_2-1)(1+\delta+\beta)}.$$

This formula implies that g is a decreasing function of g_τ, while g_{a_1} grows with g_τ. A greater growth rate in the carbon tax accelerates technical change in the energy-saving sector, but reduces long-run GDP growth.

This regime prevails if and only if the corresponding value of g_E is indeed positive, i.e.

$$g_\tau \leq \frac{\theta N}{c_2(\sigma_2 - 1) + c_1(\sigma_1 - 1)}.$$

Regime 2: $g_E < 0$

In this case the long-run value of g_Z is equal to zero, and the corresponding equations determining growth rates are

$$g_E = \frac{g - g_\tau}{1 + \beta}$$

and

$$g_{a_1} = g - g_E = \frac{g\beta + g_\tau}{1 + \beta}.$$

The growth rate is now determined by

$$\frac{\beta g + g_\tau}{1 + \beta} c_1(\sigma_1 - 1) + g c_2(\sigma_2 - 1) = \theta N,$$

implying

$$g = \frac{\theta N(1 + \beta) - g_\tau c_1(\sigma_1 - 1)}{c_1(\sigma_1 - 1)\beta + c_2(\sigma_2 - 1)(1 + \beta)},$$

$$g_E = \frac{\theta N - g_\tau(c_1(\sigma_1 - 1) + c_2(\sigma_2 - 1))}{c_1(\sigma_1 - 1)\beta + c_2(\sigma_2 - 1)(1 + \beta)}.$$

This last formula confirms that this regime prevails if and only if

$$g_\tau > \frac{\theta N}{c_2(\sigma_2 - 1) + c_1(\sigma_1 - 1)},$$

while we see that the growth rate of GDP is positive provided

$$g_\tau < \frac{\theta N(1 + \beta)}{c_1(\sigma_1 - 1)}.$$

These formulas imply that there exists a "window" of sustainable development, in terms of the growth rate tax, given by:

$$\frac{\theta N}{c_2(\sigma_2 - 1) + c_1(\sigma_1 - 1)} < g_\tau < \frac{\theta N(1+\beta)}{c_1(\sigma_1 - 1)}.$$

If g_τ satisfies that inequality, then output grows while emission fall. With the

above defined set of parameters, this window is wide: it goes from a growth rate of the carbon tax of 2.3 % a year, which yields constant emissions and thus a linearly (but not exponentially) exploding stock of CO2 in the atmosphere, and a GDP growth rate of 2.3 % a year, to a tax growing at 21.6 % a year, corresponding to a zero growth scenario. For a tax growing at 3 % a year, emissions decline at -0.6 % a year, while GDP growth is at 2.2 % a year.

12. Endogenizing the number of researchers

In the preceding analysis, the carbon tax has either no impact on long-run growth, or, if it grows, it reduces long-run growth. This runs counter to some arguments according to which it could yield a "double dividend", by both reducing emissions and boosting the "competitiveness" of business firms (for example, Porter and van den Linde, 1995). While it is not always clear what one means by "competitiveness", the above results suggest that at least the economy's overall productivity level does not go up with the (rate of growth) of the carbon tax.

Could this result be overturned? One strong assumption is that the overall number of researchers is fixed. Total R and D effort cannot go up in response to an increase in the relative income of researchers. If some supply response of researchers were allowed, *and* if a carbon tax increased the value of R and D and hence their wages, in principle it could trigger an increase in the number of researchers and therefore in long-run growth.

To examine this possibility, we have studied an extension of the model where there exists a representative consumer who allocates his time optimally between productive labor and R and D. The utility being maximized must be modified as follows:

$$\max \int \left[\frac{C_t^{\gamma}}{\gamma} + \zeta \left((1-\theta_t)^{\eta} + \varepsilon \theta_t^{\eta} \right)^{1/\eta} \right] e^{-\rho t} dt$$

The fraction of researchers θ is now endogenous and time-dependent. The added term $\zeta \left((1-\theta_t)^{\eta} + \varepsilon \theta_t^{\eta} \right)^{1/\eta}$ reflects concave preferences for the allocation of time between research and production. An additional first-order condition corresponding to optimality with respect to θ must be added to the model. It is given by

$$\frac{C_t^{\gamma-1}}{\zeta} (w_t - w_{Rt}) = \left((1-\theta_t)^{\eta} + \varepsilon \theta_t^{\eta} \right)^{1/\eta - 1} \left(\varepsilon \theta_t^{\eta-1} - (1-\theta_t)^{\eta-1} \right) \qquad (12.1)$$

Finally one can show that a balanced growth path with a constant value of θ exists provided $\gamma = 0$, i.e. for a logarithmic utility function. Hence endogenizing the number of researchers has a cost, in that to get a BGP utility must be logarithmic in consumption. Note, however, that this is not far-off usual parameters in the macro literature.

While it is not possible to get an analytical solution for the BGP, it can be computed numerically by iteration on θ until (12.1) is met.

What do we find? Our numerical simulations have strikingly different implications depending on whether one considers a carbon tax or a subsidy to green R and D.

In the case of a carbon tax, the model implies that it has zero effects on long-run growth even though the number of researchers is endogenous. That number simply does not respond to the carbon tax. The extension is therefore useless in that case. The numerical simulations suggest that the effect is truly nil, rather than negligible, which seems to come from some underlying analytical property of the model.

In the case of a subsidy to green R and D, our simulations imply that it has a strong positive effect on long-run growth. This is not surprising since it is a subsidy to R and D, albeit directed. The following Table gives the effect of the green R and D subsidy on the fraction of researchers in the economy and the long-run growth rate starting from a benchmark simulation where $\theta \approx 0.02$. That simulation was reached by setting $\eta = 0.8$ and $\varepsilon = 0.439$; ζ was normalized to 1. For lower elasticities of the supply of researchers, like $\eta = 0.2$, we get some 0.1 percentage points of growth for 10 percentage points more of subsidy.

Table 7. Impact of a subsidy to green R & D on growth

s_1	θ	g
0	0.0196	2.35 %
0.1	0.0218	2.62 %
0.2	0.0241	2.89 %

The message of this section is therefore as follows: There is no double-dividend, in that a carbon tax does not accelerate growth; but there is a "green way" of subsidizing R and D, which is efficient in boosting growth. Recall, however, that given θ, the green subsidy only has a level effect on emissions, while g_E goes up with θ. Thus the "green subsidy" boosts growth, but also the growth rate of emissions. It does not achieve sustainable development; simply, the level of emissions is lower than for an equivalent overall subsidy to R and D. Consequently, from the point of view of emission reduction, the additional growth permitted by the green subsidy may be undesirable. Presumably one would like to offset it by an exploding carbon tax or by a tax on nongreen R and D.

13. Conclusion

This paper has described the basic theoretical block of the Toulouse model of endogenous growth and carbon emissions. In the future it is planned to study extensions of this model with renewable energies and also a multi-region version of the model, while a numerical procedure is being developed for solving the model's transitional dynamics.

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APPENDIX

COMPUTING LEVELS IN THE STEADY STATE.

One can check that the following procedure delivers a balanced growth path as a solution to the model.

1. The interest rate is constant and equal to

$$r = \rho + g_{a_2}/\sigma$$

2. The equilibrium value of μ

$$\mu = \frac{g_{a_1} c_1(\sigma_1 - 1)}{N\theta}$$

3. Using equation (9.5), one must have

$$Z_0 = E_0/g_E$$

4. Using (9.1)-(9.2), (9.8)-(9.9), (9.10), (9.11) and (9.6),we get a block of three equations allowing to get w_0, q_0, E_0 as a function of a_{20} :

$$(1-\theta)N\frac{a_{20}}{a_{10}E_0} = \left(\frac{b_2}{b_1}\frac{q_0}{a_{10}}\frac{a_{20}}{w_0}\frac{\sigma_2 - 1}{\sigma_2}\frac{\sigma_1}{\sigma_1 - 1}\right)^{1/(1-\alpha)}$$

$$1 = b_1^{1/(1-\alpha)}\left(\frac{q_0}{a_{10}}\frac{\sigma_1}{\sigma_1 - 1}\right)^{-\alpha/(1-\alpha)} + b_2^{1/(1-\alpha)}\left(\frac{w_0}{a_{20}}\frac{\sigma_2}{\sigma_2 - 1}\right)^{-\alpha/(1-\alpha)}$$

$$q_0/(1+\tau) = (1+\beta)\phi g_E^{-\delta} E_0^{\delta+\beta}$$

This block can be solved recursively. One gets

$$q_0 = a_{10}\lambda\left[a_{20}(1+\tau)^{1/(\delta+\beta)}a_{10}^{-(1+1/(\delta+\beta))}\right] \qquad (13.1)$$

$$w_0 = \left[\frac{1}{\phi(1+\tau_0)(1+\beta)}\right]^{\frac{1-\alpha}{\delta+\beta}} q_0^{1+\frac{1-\alpha}{\delta+\beta}} g_E^{\frac{\delta(1-\alpha)}{\delta+\beta}} \left((1-\theta)\,N\right)^{-(1-\alpha)} \left(\frac{a_{20}}{a_{10}}\right)^{\alpha} \left(\frac{b_2}{b_1}\frac{\sigma_2 - 1}{\sigma_2}\frac{\sigma_1}{\sigma_1 - 1}\right)$$

$$E_0 = (1-\theta)N\left(\frac{a_{20}}{a_{10}}\right)^{-\frac{\alpha}{1-\alpha}}\left(\frac{q_0}{w_0}\frac{b_2}{b_1}\frac{\sigma_2-1}{\sigma_2}\frac{\sigma_1}{\sigma_1-1}\right)^{-1/(1-\alpha)},$$

where the $\lambda(.)$ function is defined so that $\lambda(y)$ is the value of x solution to

$$\kappa_0 x^{-\alpha/(1-\alpha)} + \kappa_1 x^{-\frac{\alpha}{1-\alpha}-\frac{\alpha}{\delta+\beta}} y^\alpha = 1,$$

where

$$\begin{aligned}
\kappa_0 &= b_1^{\frac{1}{1-\alpha}}\left(\frac{\sigma_1}{\sigma_1-1}\right)^{-\frac{\alpha}{1-\alpha}} \\
\kappa_1 &= g_E^{-\frac{\alpha\delta}{\delta+\beta}}((1-\theta)N)^\alpha(\phi(1+\beta))^{\frac{\alpha}{\delta+\beta}} b_2 b_1^{\frac{\alpha}{1-\alpha}}\left(\frac{\sigma_1}{\sigma_1-1}\right)^{-\frac{\alpha}{1-\alpha}}
\end{aligned}$$

Hence, these steps allow to compute w_0, q_0, E_0 as a function of a_{20}, but one has to compute the right value of a_{20}.

5. One can also compute, as a function of a_{20} :

$$X_{10} = a_{10}E_0$$

$$X_{20} = a_{20}N(1-\theta)$$

$$Y_0 = [b_1 X_{10}^\alpha + b_2 X_{20}^\alpha]^{1/\alpha}$$

$$C_0 = Y_0 - \phi E_0^{1+\delta+\beta} g_E^{-\delta}$$

6. Finally, one must compute the equilibrium values of a_{20}. To do so, we use the arbitrage condition between the two types of R and D, which implies that

$$\frac{V_{1t}}{c_1 a_{1t}^{1-\sigma_1}(1-s_1)} = \frac{V_{2t}}{c_2 a_{2t}^{1-\sigma_2}(1-s_2)} = w_R.$$

Given that:

$$\pi_{10} = X_{10} q_0 a_{10}^{-\sigma_1}(\sigma_1-1)^{-1}$$

$$\pi_{20} = X_{20} w_0 a_{20}^{-\sigma_2} (\sigma_2 - 1)^{-1},$$

and

$$V_{i0} = \frac{\pi_{i0}}{r - g_{\pi_i}}, \tag{13.2}$$

this is equivalent to

$$\frac{\pi_{10}}{r - g_{\pi_1}} \frac{1}{c_1(1-s_1)} a_{10}^{\sigma_1 - 1} = \frac{\pi_{20}}{r - g_{\pi_2}} \frac{1}{c_2(1-s_2)} a_{20}^{\sigma_2 - 1}.$$

using the preceding formulas, this is equivalent to:

$$\frac{E_0 q_0}{(\sigma_1 - 1) c_1 (1 - s_1)(r - g_{\pi_1})} = \frac{N(1-\theta) w_0}{(\sigma_2 - 1) c_2 (1 - s_2)(r - g_{\pi_2})}$$

Using (13.1)-(**??**) and other equations we can rewrite that as an equation determining a_{20} as a function of a_{10} :

$$\begin{aligned} & \frac{1}{\phi(1+\beta)(1+\tau_0)} a_{10}^{1+\delta+\beta} \lambda \left[a_{20} (1+\tau_0)^{1/(\delta+\beta)} a_{10}^{-(1+1/(\delta+\beta))} \right] a_{20}^{-(\delta+\beta)} \\ = \quad & (N(1-\theta))^{\delta+\beta} \left(\frac{\sigma_1 c_1 (1-s_1)(r-g_{\pi_1})}{\sigma_2 c_2 (1-s_2)(r-g_{\pi_2})} \right)^{\frac{\delta+\beta}{\alpha}} g_E^{-\delta} \left(\frac{b_2}{b_1} \frac{\sigma_2 - 1}{\sigma_2} \frac{\sigma_1}{\sigma_1 - 1} \right)^{\frac{\delta+\beta}{\alpha}} \end{aligned}$$

Steps 1-6 therefore allow in principle to compute the initial values of each endogenous variable compatible with a BGP.

BASIC Source code for numerical computation of the steady state

```
10 rem parameters
20 al=0.3
30 b1=0.1
40 b2=0.97
50 k=0.74
60 phi=1.51
70 tau0=0.1
80 si=1:si1=2:si2=2:ro=0.01:th=0.02:nn=20
90 c1=2.5:c2=15
100 de=1.64:be=0.35
110 a10=1
1000 rem computes growth rates
1010 ga1 = (de+be)*th*nn/(c1*(si1-1)*(de+be)+c2*(si2-1)*(1+de+be))
1020 gq=ga1:print "ga1=gq=",gq
1030 ge = ga1/(de+be):print "ge=",ge
1040 ga2 = ga1/(de+be)*(1+de+be)
1050 gpi1 = (2-si1)*ga1+ge
1060 gpi2 = (2-si2)*ga2
1070 gy = ga2:print "gy=ga2=gc=gw=",ga2
1080 gc = ga2
1090 gw = ga2
1100 gwr = gpi1+(si1-1)*ga1:print "gwr=",gwr
1200 rem computes intermediate parameters
1210 kap0=b1^(1/(1-al))
1220 kap1=ge^(-al*de/(de+be))*((1-th)*nn)^al*(phi/k)^(al/(de+be))*b2*b1^(al/(1-al))
1230 factor=(1+tau0)^(1/(de+be))*a10^(-(1+1/(de+be)))
2000 rem computes levels as a function of a20
2010 r = ro+ga2/si:print "r=",r
```

```
2020 mu = ga1*c1*(si1-1)/nn/th:print "mu=",mu
2030 gosub 15000:print "a20=",a20
2040 y=a20*factor:gosub 10000
2050 q0=a10*lambda:print "q0=",q0
2060 w0=(phi*(1+tau0)/k)^(-(1-al)/(de+be))*q0^(1+(1-al)/(de+be))*ge^(de*(1-al)/(de+be))*((1-th)*nn)^(-(1-al))*(a20/a10)^(al)*(b2/b1):print "w0=",w0
2070 ee0=(1-th)*nn*(a20/a10)^(-al/(1-al))*(q0*b2/w0/b1)^(-1/(1-al)):print "ee0=",ee0
2080 xx10=a10*ee0:print "xx10=",xx10
2090 xx20=a20*nn*(1-th):print "xx20=",xx20
2100 yy0=(b1*xx10^al+b2*xx20^al)^(1/al):print "yy0=",yy0
2110 cc0=yy0-phi*ee0^(1+de+be)*ge^-de:print "cc0=",cc0
2120 pi10=xx10*q0*a10^-si1*si1^-si1*(si1-1)^(si1-1):print "pi10=",pi10
2130 pi20=xx20*w0*(a20^-si2)*(si2^-si2)*((si2-1)^(si2-1)):print "pi20=",pi20
2140 vv10=pi10/(r-gpi1):print "vv10=",vv10
2150 vv20=pi20/(r-gpi2):print "vv20=",vv20
2160 zz0=ee0/ge:print "zz0=",zz0
2170 qbar0=q0/(1+tau0):print "qbar0=",qbar0
2180 wr0=vv10*(si1-1)^(1-si1)/c1/si1^(1-si1)/a10^(1-si1):print "wr0=",wr0
2185 se0=q0*ee0/yy0:print "energy share=",se0
2200 stop
10000 rem this subroutine delivers lamba as a function of y
10002 rem a marche ssi delta+beta > 1-al (vrai avec ma calibration originale: de+be =2; 1-al=0.7)
10010 x=1
10020 diff=1-kap0*x^(-al/(1-al))-kap1*x^(-al/(1-al)-al/(de+be))*y^al:if diff<0 then x=2*x:goto 10020
10030 xmax=x:x=1
10040 diff=1-kap0*x^(-al/(1-al))-kap1*x^(-al/(1-al)-al/(de+be))*y^al:if diff>0 then x=x/2:goto 10040
10050 xmin=x
10060 x=(xmin+xmax)/2
```

```
10100 diff = 1-kap0*x^(-al/(1-al))-kap1*x^(-al/(1-al)-al/(de+be))*y^al
10110 if diff>0.00001 then xmax=x: goto 10060
10120 if diff<-0.00001 then xmin=x: goto 10060
10130 lambda = x: return
15000 rem this subroutine computes a20 as a function of a10, using the lambda function
15030 target = (nn*(1-th))^(de+be)*(si1*c1*(r-gpi1)/si2/c2/(r-gpi2))^((de+be)/al)*ge^-de*(b2/b1)^((de+be)/al)*phi*(1+tau0)/k/a10^(1+de+be)
15060 a20=1
15070 y=a20*factor:gosub 10000:lhs=a20^(-(de+be))*lambda:if lhs>target then a20=a20*2:goto 15070
15080 a20max=a20:a20=1
15090 y=a20*factor:gosub 10000:lhs=a20^(-(de+be))*lambda:if lhs<target then a20=a20/2:goto 15090
15100 a20min=a20
15110 a20=(a20min+a20max)/2:
15120 y=a20*factor:gosub 10000:lhs=a20^(-(de+be))*lambda
15125 if lhs<target then a20max=a20
15130 if lhs>target then a20min=a20
15140 if a20max-a20min>0.000001 then 15110
15150 return
```